

A three steps seventh order iterative method for solution nonlinear equation using Lagrange Interpolation technique

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Abstract This research paper comprehensively presents the derivation of a seventh-order iteration scheme designed to obtain simple roots of nonlinear equations through the utilization of Lagrange interpolation technique. The scheme is characterized by the requirement for three function evaluations and one evaluation of the first derivative in each iteration. A detailed convergence analysis is also carried out to assess the efficacy of the proposed method. Additionally, the paper includes comprehensive numerical experiments aimed at confirming the theoretical results and illustrating the competitive performance of the derived iteration scheme.

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1 Introduction

Numerical analysis, situated at the intersection of mathematics and computer science, focuses on the development, examination, and application of algorithms designed to solve numerical challenges within continuous mathematics. Numerous equations exist for which analytical solutions are unattainable through any established methods. In these cases, obtaining approximate roots is achieved through various ap-



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proaches, including graphical methods, iterative techniques, or a synergistic combination of both [1, 9, 12, 17, 19, 21].

In this research, we explore iterative technique for locating a single root of a nonlinear equation $\phi(\xi) = 0$, denoted as α , where $\phi(\xi) \neq 0$, but $\phi'(\xi) = 0$.

Newton's method stands out as one of the most recognized and widely employed approaches to address the nonlinear problems, as introduced by [3] and denoted by

$$\xi_{n+1} = \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \quad (1)$$

A family of fifth-order iterative methods proposed by [27] and [14], which aim to address nonlinear equations using the weight function technique. Each method in this family employs four function evaluations per iteration, achieving a consistent efficiency index (E.I) of 1.495. Another contribution by [24] presents a two-point iterative scheme designed for multiple roots, achieving fifth-order convergence through two function evaluations and two derivative evaluations per iteration, within the framework of the weight function approach. Furthermore, [25] introduces two six iterative methods, one of which is second derivative-free, for solving nonlinear equations based on Taylor series expansion and Halley's method. Both approaches exhibit an efficiency index of 1.565. In [23], a numerical iterative method within the family of open methods is developed, attaining sixth-order convergence inspired by the Steffensen Method and the Newton-Raphson Method. Additionally, two iterative methods based on Newton's method are presented by [2] with convergence orders of seven and twelve, respectively. Comparative analyses reveal the superior efficiency of these schemes over certain recent methods. In a separate contribution [4], two modified seventh-order convergent Newton-type methods are introduced, requiring three function evaluations and two derivative evaluations per iteration, yielding an efficiency index of 1.476.

Another family of iterative methods, featuring seventh and eighth-order Newton-type schemes, is proposed by [28] for solving nonlinear equations. Applications to diverse problems and comparisons with established methods are conducted to assess their effectiveness. [16] suggests two innovative iterative methods for solving nonlinear equations, demonstrating seventh and eighth-order convergence and requiring only four function evaluations per iteration, aligning with the Kung-Traub conjecture on optimal order for schemes without memory. Moreover, [26] presents three iterative methods of orders three, six, and seven for solving nonlinear equations. These methods are formulated using the modified homotopy perturbation technique along with a system of equations. Finally, [15] introduces two novel algorithms for root-finding in nonlinear scalar equations, with the second algorithm demonstrating fourth- and sixth-order convergence, identified as optimal according to Kung-Traub's conjecture. The effectiveness of these algorithms is illustrated through applications to real-life problems in engineering sciences.

The efficiency index $I.E$ is defined as $a^{1/b}$, Where a represents the order of convergence of the method and b denotes the number of functional evaluations used per iteration [10].

2 Derivation of proposed method

[6] mentioned a two-step fourth order optimal method with four function evaluation (two function and two first derivative).

i.e.

$$\begin{aligned} \text{Step.1 } \psi_n &= \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \\ \text{Step.2 } \xi_{n+1} &= \psi_n - \left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)} \right)^2 \frac{\phi(\psi_n)}{\phi'(\xi_n)} \end{aligned} \quad (2)$$

To enhance the order of the above method we add one more step of newton Raphson method as third step of (2)

$$\begin{aligned} \text{Step.1 } \psi_n &= \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \\ \text{Step.2 } \kappa_n &= \psi_n - \left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)} \right)^2 \frac{\phi(\psi_n)}{\phi'(\xi_n)} \\ \text{Step.3 } \xi_{n+1} &= \kappa_n - \frac{\phi(\kappa_n)}{\phi'(\kappa_n)} \end{aligned} \quad (3)$$

Now to increase the efficiency index, we have to reduce the number of function evaluations. So here we introduce the Lagrange's interpolation technique [5] to replace the $\phi'(\kappa_n)$.

$$L(t) = \frac{(t - \psi_n)(t - \kappa_n)}{(\xi_n - \psi_n)(\xi_n - \kappa_n)} \phi(\xi_n) + \frac{(t - \xi_n)(t - \kappa_n)}{(\psi_n - \xi_n)(\psi_n - \kappa_n)} \phi(\psi_n) + \frac{(t - \psi_n)(t - \xi_n)}{(\kappa_n - \psi_n)(\kappa_n - \xi_n)} \phi(\kappa_n) \quad (4)$$

Differentiate (4) with respect to t .

$$L'(t) = \frac{2t - \psi_n - \kappa_n}{(\xi_n - \psi_n)(\xi_n - \kappa_n)} \phi(\xi_n) + \frac{2t - \xi_n - \kappa_n}{(\psi_n - \xi_n)(\psi_n - \kappa_n)} \phi(\psi_n) + \frac{2t - \xi_n - \psi_n}{(\kappa_n - \psi_n)(\kappa_n - \xi_n)} \phi(\kappa_n) \quad (5)$$

And put $t = \kappa_n$.

$$L'(\kappa_n) = \frac{(\kappa_n - \psi_n)}{(\xi_n - \psi_n)(\xi_n - \kappa_n)} \phi(\xi_n) + \frac{(\kappa_n - \xi_n)}{(\psi_n - \xi_n)(\psi_n - \kappa_n)} \phi(\psi_n) + \frac{2\kappa_n - \xi_n - \psi_n}{(\kappa_n - \psi_n)(\kappa_n - \xi_n)} \phi(\kappa_n) \quad (6)$$

after simplify above equation, we got

$$L'(\kappa_n) = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n} + \frac{\phi(\kappa_n) - \phi(\psi_n)}{\kappa_n - \psi_n} - \frac{\phi(\psi_n) - \phi(\xi_n)}{\psi_n - \xi_n} \quad (7)$$

$$\text{or } L'(\kappa_n) = \phi[\xi_n, \kappa_n] + \phi[\psi_n, \kappa_n] - \phi[\xi_n, \psi_n] \quad (8)$$

Now replace $\phi(\kappa_n)$ by $L'(\kappa_n)$ in (3). And finally our proposed scheme will be

$$\begin{aligned} \text{Step.1 } \psi_n &= \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} \\ \text{Step.2 } \kappa_n &= \psi_n - \left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)} \right)^2 \frac{\phi(\psi_n)}{\phi'(\xi_n)} \\ \text{Step.3 } \xi_{n+1} &= \kappa_n - \frac{\phi(\kappa_n)}{\phi[\xi_n, \kappa_n] + \phi[\psi_n, \kappa_n] - \phi[\xi_n, \psi_n]} \end{aligned} \quad (9)$$

Equation (9) is our proposed three step scheme, used four function evaluations (three function and one first derivative).

3 Convergence Analysis

Theorem I: Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $\phi: D \subset \mathbb{R} \rightarrow \mathbb{R}$ in an open interval D , which contains ξ_0 as an initial approximation of α . Then the method (9) is of order seventh and includes only three function evaluations and one first derivative per full iteration, and no second or higher derivatives are used.

Proof.

We write down the Taylor's series expansion of the function $\phi(\xi_n)$.

$$\phi(\xi_n) = \sum_{m=0}^{\infty} \frac{\phi^{(m)}(\alpha)}{m!} (\xi_n - \alpha)^m = \phi(\alpha) + \phi'(\alpha)(\xi_n - \alpha) + \frac{\phi''(\alpha)}{2!} (\xi_n - \alpha)^2 + \frac{\phi'''(\alpha)}{3!} (\xi_n - \alpha)^3 + \dots \quad (10)$$

For simplicity, we assume that

$$A_k = \left(\frac{1}{k!} \right) \frac{\phi^{(k)}(\alpha)}{\phi'(\alpha)}, \quad k \geq 2.$$

And assume that $\delta_n = \xi_n - \alpha$. Thus, we have

$$\phi(\xi_n) = \phi'(\alpha) \left[\delta_n + A_2 \delta_n^2 + A_3 \delta_n^3 + A_4 \delta_n^4 + \dots + O(\delta_n^8) \right] \quad (11)$$

Furthermore, we have

$$\phi'(\xi_n) = \phi'(\alpha) \left[1 + 2A_2 \delta_n + 3A_3 \delta_n^2 + 4A_4 \delta_n^3 + \dots + O(\delta_n^8) \right] \quad (12)$$

And hence, we have

$$\text{Step 1. } \psi_n = \xi_n - \frac{\phi(\xi_n)}{\phi'(\xi_n)} = \delta_n - \frac{\phi'(\alpha) \left[\delta_n + A_2 \delta_n^2 + A_3 \delta_n^3 + A_4 \delta_n^4 + \dots + O(\delta_n^8) \right]}{\phi'(\alpha) \left[1 + 2A_2 \delta_n + 3A_3 \delta_n^2 + 4A_4 \delta_n^3 + \dots + O(\delta_n^8) \right]} \quad (13)$$

$$\psi_n = A_2 \delta_n^2 + (2A_3 - 2A_2^2) \delta_n^3 + (4A_2^3 - 7A_2 A_3 + 3A_4) \delta_n^4 + \dots + O(\delta_n^8) \quad (14)$$

Again expanding $\phi(\psi_n)$ about α , we have

$$\phi(\psi_n) = \phi'(\alpha) [\psi_n + A_2 \psi_n^2 + A_3 \psi_n^3 + A_4 \psi_n^4 + \dots + O(\psi_n^8)] \quad (15)$$

And then from (15), we have

$$\phi(\psi_n) = \phi'(\alpha) [A_2 \delta_n^2 + (2A_3 - 2A_2^2) \delta_n^3 + (5A_2^3 - 7A_2 A_3 + 3A_4) \delta_n^4 + \dots + O(\delta_n^8)] \quad (16)$$

From equation (11) and (16) we got

$$\frac{\phi(\psi_n)}{\phi'(\xi_n)} = A_2 \delta_n^2 + (2A_3 - 4A_2^2) \delta_n^3 + (13A_2^3 - 14A_2 A_3 + 3A_4) \delta_n^4 + \dots + O(\delta_n^8) \quad (17)$$

And

$$\phi(\xi_n) + 2\phi(\psi_n) = \phi'(\alpha) \left(\delta_n - A_2 \delta_n^2 + (4A_2^2 - 3A_3) \delta_n^3 + (-10A_2^3 + 14A_2 A_3 - 5A_4) \delta_n^4 + \dots + O(\delta_n^8) \right) \quad (18)$$

and

$$\phi(\xi_n) + \phi(\psi_n) = \phi'(\alpha) \left(\delta_n + (2A_2^2 - A_3) \delta_n^3 + (-5A_2^3 + 7A_2 A_3 - 2A_4) \delta_n^4 + \dots + O(\delta_n^8) \right) \quad (19)$$

form equation (18) and (19), we have

$$\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)} = 1 + A_2\delta_n + 2(-2A_2^2 + A_3)\delta_n^2 + (15A_2^3 - 14A_2A_3)\delta_n^3 + (-55A_2^4 + 75A_2^2A_3 - 12A_3^2)\delta_n^4 + \dots + O(\delta_n^8) \quad (20)$$

Taking square of above equation

$$\left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)}\right)^2 = 1 + 2A_2\delta_n + (-7A_2^2 + 4A_3)\delta_n^2 + 2(11A_2^3 - 12A_2A_3)\delta_n^3 - 2(32A_2^4 - 53A_2^2A_3 + 10A_3^2)\delta_n^4 + \dots + O(\delta_n^8) \quad (21)$$

and

$$\left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)}\right)^2 \left(\frac{\phi(\psi_n)}{\phi'(\xi_n)}\right) = A_2\delta_n^2 + (-3A_2^2 + 2A_3)\delta_n^3 + 5(A_2^3 - 2A_2A_3)\delta_n^4 + 2(3A_2^4 + 10A_2^2A_3 - 4A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \quad (22)$$

Then step 2 will be

$$\text{Step 2 } \kappa_n = \psi_n - \left(\frac{\phi(\xi_n) + 2\phi(\psi_n)}{\phi(\xi_n) + \phi(\psi_n)}\right)^2 \left(\frac{\phi(\psi_n)}{\phi'(\xi_n)}\right) = (6A_2^3 - A_2A_3)\delta_n^4 - 2(23A_2^4 - 19A_2^2A_3 + A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \quad (23)$$

and

$$\phi(\kappa_n) = \phi'(\alpha) \left((6A_2^3 - A_2A_3)\delta_n^4 - 2(23A_2^4 - 19A_2^2A_3 + A_3^2)\delta_n^5 + 3(77A_2^5 - 112A_2^3A_3 + 26A_2A_3^2)\delta_n^6 + \dots + O(\delta_n^8) \right) \quad (24)$$

from (11) and (24), we have

$$\phi(\kappa_n) - \phi(\xi_n) = \phi'(\alpha) \left(-\delta_n - A_2\delta_n^2 - A_3\delta_n^3 + (6A_2^3 - A_2A_3)\delta_n^4 - 2(23A_2^4 - 19A_2^2A_3 + A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \right) \quad (25)$$

and

$$\phi[\xi_n, \kappa_n] = \frac{\phi(\kappa_n) - \phi(\xi_n)}{\kappa_n - \xi_n} = \phi'(\alpha) \left(1 + A_2\delta_n + A_3\delta_n^2 + (6A_2^4 - A_2^2A_3)\delta_n^4 + (-46A_2^5 + 44A_2^3A_3 - 3A_2A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \right) \quad (26)$$

from (16) and (24), we have

$$\phi(\kappa_n) - \phi(\psi_n) = \phi'(\alpha) \left(-A_2\delta_n^2 + 2(A_2^2 - A_3)\delta_n^3 + (A_2^3 + 6A_2A_3)\delta_n^4 + 2(-17A_2^4 + 7A_2^2A_3 + 2A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \right) \quad (27)$$

and

$$\phi[\psi_n, \kappa_n] = \frac{\phi(\kappa_n) - \phi(\psi_n)}{\kappa_n - \psi_n} = \phi'(\alpha) \left(1 - A_2^2\delta_n^2 + 2A_2(A_2^2 - A_3)\delta_n^3 + A_2^2(10A_2^2 - 7A_3)\delta_n^4 + \dots + O(\delta_n^8) \right) \quad (28)$$

from (11) and (16), we have

$$\phi(\psi_n) - \phi(\xi_n) = \phi'(\alpha) \left(-\delta_n + (-2A_2^2 + A_3)\delta_n^3 + (5A_2^3 - 7A_2A_3)\delta_n^4 - 6(2A_2^4 - 4A_2^2A_3 + A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \right) \quad (29)$$

$$\phi[\xi_n, \psi_n] = \frac{\phi(\psi_n) - \phi(\xi_n)}{\psi_n - \xi_n} = \phi'(\alpha) \left(1 - A_2\delta_n + (A_2^2 + A_3)\delta_n^2 + (-2A_2^3 + 3A_2A_3)\delta_n^3 + 2(2A_2^4 - 4A_2^2A_3 + A_3^2)\delta_n^4 + \dots + O(\delta_n^8) \right) \quad (30)$$

from (26), (28) and (30), we have

$$\phi[\xi_n, \kappa_n] + \phi[\psi_n, \kappa_n] - \phi[\xi_n, \psi_n] = \phi'(\alpha) \left(1 - A_2A_3\delta_n^3 + 2(6A_2^4 + A_3^2)\delta_n^4 - 2A_2(46A_2^4 - 39A_2^2A_3 - A_3^2)\delta_n^5 + \dots + O(\delta_n^8) \right) \quad (31)$$

from (24) and (31), we have

$$\frac{\phi(\kappa_n)}{\phi[\xi_n, \kappa_n] + \phi[\psi_n, \kappa_n] - \phi[\xi_n, \psi_n]} = (6A_2^3 - A_2A_3) \delta_n^4 - 2(23A_2^4 - 19A_2^2A_3 + A_3^2) \delta_n^5 + \dots + O(\delta_n^8) \quad (32)$$

Finally

$$\text{Step 3 } \xi_{n+1} = \kappa_n - \frac{\phi(\kappa_n)}{\phi[\xi_n, \kappa_n] + \phi[\psi_n, \kappa_n] - \phi[\xi_n, \psi_n]} = A_2A_3 (A_2A_3 - 6A_2^3) \delta_n^7 + \dots + O(\delta_n^8) \quad (33)$$

Finally, the rate of convergence of proposed scheme (9) is seven, and efficiency index is 1.627.

4 Numerical Experiments

These problems taken from references See in [1–28].

Maple 2022 and Origin 2021 software were utilized for numerical calculations and line plots, respectively. The outcomes were recorded using an Intel(R) Core(TM) i3-4010U laptop with 4.00GB of RAM and a processing speed of 1.70GHz.

| S.No | Problems | Initial guess | Exact root upto 15dp |
|------|---|----------------|----------------------------|
| 1 | $\phi(\xi) = \xi^3 + 4\xi^2 - 10$ | $\xi_0 = 1.5$ | $\xi = 1.365230013414097$ |
| 2 | $\phi(\xi) = e^{-\xi^2 + \xi + 2} - 1$ | $\xi_0 = -1.9$ | $\xi = -1.000000000000000$ |
| 3 | $\phi(\xi) = \log(\xi^2 + \xi + 2) - 1$ | $\xi_0 = 9.9$ | $\xi = -1.484013124129473$ |
| 4 | $\phi(\xi) = e^\xi + \cos(\xi) - 1$ | $\xi_0 = 1$ | $\xi = -0.923632658955135$ |
| 5 | $\phi(\xi) = \xi^2 - e^\xi - 3\xi + 2$ | $\xi_0 = -2.6$ | $\xi = -0.254280675772883$ |
| 6 | $\phi(\xi) = \xi^2 + \sin(\xi/5) + 1/4$ | $\xi_0 = 0.5$ | $\xi = 4.099920179891371$ |

Table 1. Absolute functional values of example 1, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|------------------|------------------|------------------|------------------|
| 1 | $2.22204E - 8$ | $1.34345E - 1$ | $2.10213E - 7$ | $3.92040E - 7$ |
| 2 | $2.65218E - 63$ | $5.28461E - 4$ | $2.01454E - 56$ | $2.67807E - 54$ |
| 3 | $9.15294E - 448$ | $8.29055E - 9$ | $1.49549E - 399$ | $1.85893E - 384$ |
| 4 | | $2.04055E - 18$ | | |
| 5 | | $1.23616E - 37$ | | |
| 6 | | $4.53662E - 76$ | | |
| 7 | | $6.11007E - 153$ | | |
| 8 | | $1.10834E - 306$ | | |

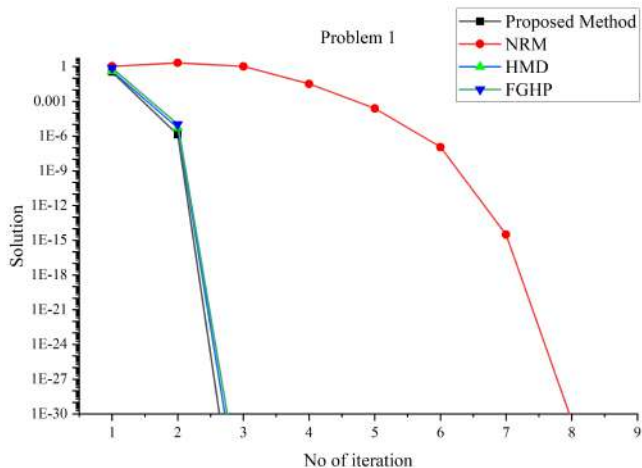


Figure 1. Graphical Representation of absolute functional values of table 1. by supposing $E - 10 = E - 1$

Table 2. Absolute functional values of example 2, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|-----------------|----------------|----------------|----------------|
| 1 | 1.58464E - 7 | 3.06406E - 2 | 6.52285E - 7 | 1.31513E - 6 |
| 2 | 1.79581E - 50 | 3.51405E - 4 | 2.51811E - 46 | 6.19636E - 44 |
| 3 | 4.31095E - 351 | 4.80007E - 8 | 3.21764E - 322 | 3.19376E - 305 |
| 4 | | 8.96026E - 16 | | |
| 5 | | 3.12224E - 32 | | |
| 6 | | 3.79105E - 62 | | |
| 7 | | 5.58912E - 123 | | |
| 8 | | 1.21482E - 247 | | |
| 9 | | 5.73920E - 495 | | |

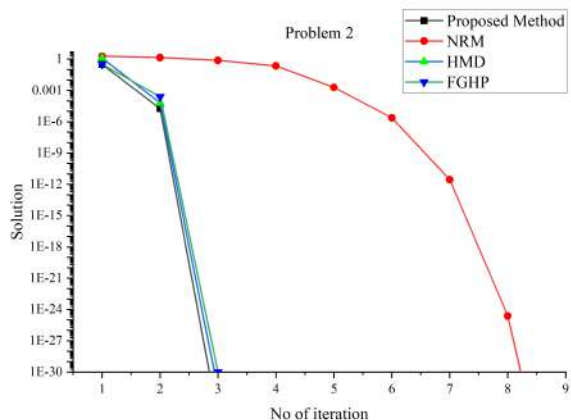


Figure 2. Graphical Representation of absolute functional values of table 2. by supposing $E - 10 = E - 1$

Table 3. Absolute functional values of example 3, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|------------------|-----------------|----------------|----------------|
| 1 | $1.58006E + 0$ | $3.44809E + 0$ | $4.75176E + 0$ | $1.05321E + 1$ |
| 2 | $8.81747E - 2$ | $2.91430E + 0$ | $9.44929E + 0$ | $1.63630E + 1$ |
| 3 | $1.13948E - 10$ | $1.66188E + 0$ | $2.03591E + 1$ | $2.46467E + 1$ |
| 4 | $2.29494E - 72$ | $4.20100E - 1$ | $3.67965E + 1$ | $3.50207E + 1$ |
| 5 | $3.08464E - 504$ | $8.65654E - 1$ | $5.73511E + 1$ | $4.70214E + 1$ |
| 6 | | $7.26556E - 2$ | $8.10195E + 1$ | $6.03120E + 1$ |
| 7 | | $1.47238E - 3$ | $1.07143E + 2$ | $7.46583E + 1$ |
| 8 | | $4.34684E - 7$ | $1.35275E + 2$ | $8.98930E + 1$ |
| 9 | | $3.81360E - 14$ | $1.65097E + 2$ | $1.05892E + 2$ |
| 10 | | $2.93534E - 28$ | $1.96373E + 2$ | $1.22560E + 2$ |

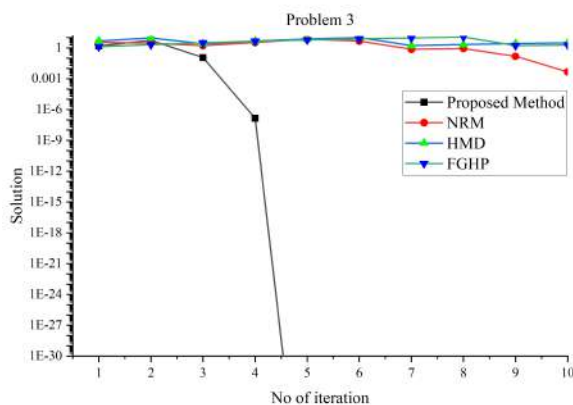


Figure 3. Graphical Representation of absolute functional values of table 3. by supposing $E - 10 = E - 1$

Table 4. Absolute functional values of example 4, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|-------------------|------------------|------------------|------------------|
| 1 | $7.11688E - 3$ | $7.95321E - 1$ | $1.75329E - 2$ | $3.20038E - 2$ |
| 2 | $1.29638E - 20$ | $7.33465E - 2$ | $1.01601E - 18$ | $5.67485E - 17$ |
| 3 | $8.80606E - 145$ | $3.48380E - 4$ | $2.08644E - 132$ | $2.80240E - 120$ |
| 4 | $5.87657E - 1014$ | $8.74545E - 9$ | $3.21346E - 928$ | $2.00706E - 843$ |
| 5 | | $5.51376E - 18$ | | |
| 6 | | $2.19168E - 36$ | | |
| 7 | | $3.46289E - 73$ | | |
| 8 | | $8.64493E - 147$ | | |
| 9 | | $5.38773E - 294$ | | |
| 10 | | $2.09264E - 588$ | | |

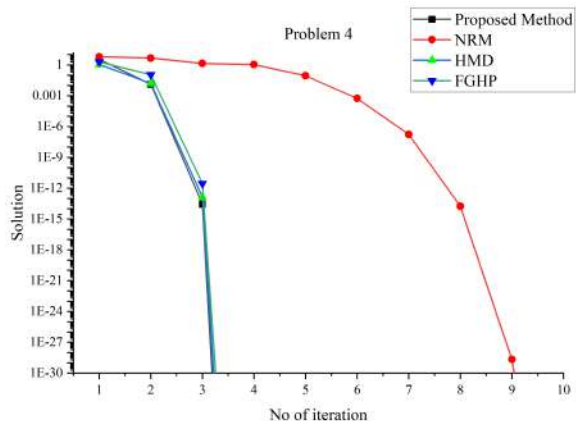


Figure 4. Graphical Representation of absolute functional values of table 4. by supposing $E - 10 = E - 1$

Table 5. Absolute functional values of example 5, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|-------------------|------------------|-------------------|-------------------|
| 1 | $4.94623E - 3$ | $3.64728E + 0$ | $1.22829E - 2$ | $3.54371E - 2$ |
| 2 | $5.97885E - 26$ | $3.77219E - 1$ | $7.56993E - 23$ | $5.72089E - 20$ |
| 3 | $2.17896E - 186$ | $3.77746E - 3$ | $2.54290E - 164$ | $1.62193E - 144$ |
| 4 | $1.86070E - 1309$ | $3.53207E - 7$ | $1.22743E - 1154$ | $2.38788E - 1016$ |
| 5 | | $3.08547E - 13$ | | |
| 6 | | $2.35453E - 31$ | | |
| 7 | | $1.37111E - 63$ | | |
| 8 | | $4.64950E - 128$ | | |
| 9 | | $5.34656E - 257$ | | |
| 10 | | $7.06987E - 515$ | | |

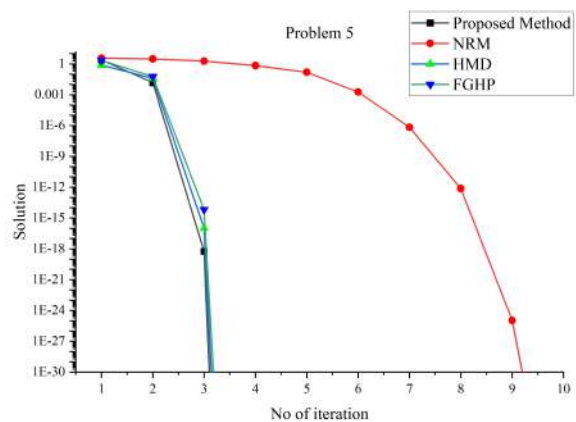


Figure 5. Graphical Representation of absolute functional values of table 5. by supposing $E - 10 = E - 1$

Table 6. Absolute functional values of example 6, tolerance fixed at $\delta = E - 300$.

| No of Iterations | PROPOSED METHOD | NRM | HMD | FGHP |
|------------------|------------------|------------------|------------------|------------------|
| 1 | $4.02334E - 8$ | $6.91979E - 3$ | $4.57366E - 8$ | $7.81550E - 8$ |
| 2 | $1.09567E - 54$ | $4.48180E - 5$ | $6.59726E - 52$ | $4.20783E - 50$ |
| 3 | $1.21690E - 380$ | $1.92976E - 9$ | $8.57157E - 359$ | $5.51782E - 346$ |
| 4 | | $3.57833E - 18$ | | |
| 5 | | $1.23036E - 35$ | | |
| 6 | | $1.45459E - 70$ | | |
| 7 | | $2.03308E - 140$ | | |
| 8 | | $3.97175E - 280$ | | |
| 9 | | $1.51579E - 559$ | | |

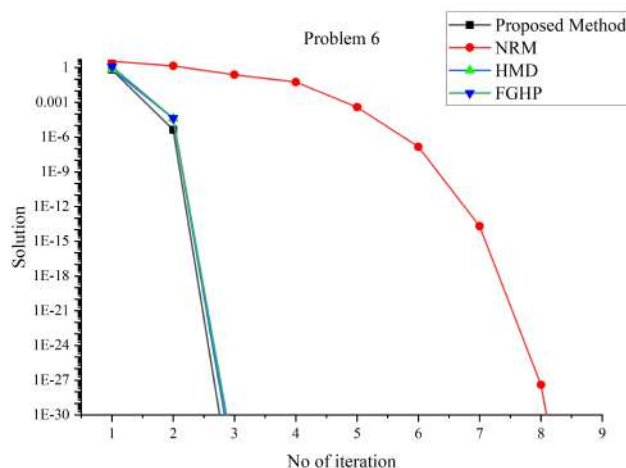


Figure 6. Graphical Representation of absolute functional values of table 6. by supposing $E - 10 = E - 1$

The comparison table 7. illustrates the Scheme, Order of Scheme, Function Evaluations (FE), Efficiency Index (EI), and Function Evaluations Per Iteration. In the last column, the coefficient of n represents the number of function evaluations, while the coefficient of n^2 indicates the number of first-order derivatives required for each iteration. An examination of the table reveals that the proposed scheme exhibits greater efficiency compared to all other methods.

Table 7. Theoretical Evaluation of Contemplated Schemes: A Comparative Analysis.

| Scheme | Order | FE | EI | Function Evaluations Per Iteration, $n > 1$ |
|-----------------|-------|----|-------|---|
| NRM [3] | 2 | 2 | 1.414 | $n + n^2$ |
| HMD [2] | 7 | 5 | 1.476 | $3n + 2n^2$ |
| FGHP [4] | 7 | 5 | 1.476 | $3n + 2n^2$ |
| Proposed scheme | 7 | 4 | 1.627 | $3n + n^2$ |

5 Conclusion

This research introduces a seventh-order, three-step method utilizing Lagrange's interpolation technique. The proposed method, requiring only four function evaluations per iteration, has demonstrated superior convergence performance compared to its counterparts. The findings, as presented in Table 1, 2, 3, 4, 5 and Figure 1, 2, 3, 4, 5, and 6, consistently show that the proposed method converges faster across all test problems.

Notably, the comparison with HMD and FGHP reveals divergence in problem 3, as indicated in Table 3. and Figure 3. In contrast, the proposed method exhibits convergence across all test problems, highlighting its reliability. The efficiency index further supports the effectiveness of the proposed method, with an impressive value of 1.627 as shown in table 7. In comparison table 1. shows that NRM, HMD, and FGHP have efficiency indices of 1.414, 1.476, and 1.476, respectively and utilizing a higher number of functional evaluations as compared to proposed method. This emphasizes that the proposed method not only outperforms in convergence but also demonstrates a higher efficiency index compared to its counterparts.

6 Future Work

To apply the above proposed seventh order scheme to the system of non-linear equations and to develop their basin of attraction.

Credit Author Statement

Fareed Ahmed Lakho: Solution of the problem, Investigation, Methodology, **Zubair Ahmed Kalhoro:** Supervision. Conceptualization, **Sanallah Jamali:** Visualization, Software support, and use latex Software, **Abdul Wasim Shaikh:** Software support, and **Jinrui Guan:** use latex Software.

Compliance with Ethical Standards

The authors declare no conflict of interest. Furthermore, informed consent was obtained from all individual participants included in the study.

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