VFAST Transactions on Mathematics
http://vfast.org/journals/index.php/VTM@ 2024 ISSN(e): 2309-0022, ISSN(p): 2411-6343
Volume 12, Number 1, January-June, 2024 pp: 60-70

Exact Solution of Tank Drainage for Bingham Fluid flow in circular tube

Ghulam Murtaza Oghai[1*], **Syed Feroz Shah**[1], **K. N. Memon**[2], **Azam Ali Amur**[3]

[1]Department of BSRS, Mehran University of Engineering and Technology Jamshoro, Pakistan; [2]Department of Mathematics and Statistics, QUEST, Nawabshah, Pakistan; [3]GBLSS Soomar khaskheli, School Education and literacy department Government of sindh

Keywords: *Tank Drainage Pproblem, Bingham Plastic Fluid, Partial Differential Equations.* ***Subject Classification***: *separation of variables Method, Modeling for nonlinear partial differential equation.*

Journal Info:
Submitted: February 27, 2024
Accepted: March 12, 2024
Published: March 23, 2024

Abstract In this paper, unstable cylindrical tank drainage of Bingham Plastic fluid with gravity-driven flow is investigated. The fluid's stress and velocity distribution are derived under no-slip conditions. The fundamental purpose of this study is to develop a mathematical model for calculating the velocity distribution of Bingham fluid, shear stress on the pipe, and other relevant discoveries. It has been shown that increasing the Bingham model parameters, flow area, and pipe length increases the time necessary to drain fluid from the tank. Furthermore, our findings show that increased density and a bigger pipe radius lead to faster drainage of Bingham plastic fluid from the circular tank.

***Correspondence Author Email Address:**
oghai811@gmail.com
DOI: 10.21015/vtm.v12i1.1737

1 Introduction

A fluid is any material that deforms continuously under the action of shearing force." A Fluid is classified into main three classes as Newtonian, Non-Newtonian and in-viscid fluid: viscid, it is viscous and sticky, which provides great resistance to motion, Newtonian which possess a constant viscosity no matter the applied stress or shear rate and non-Newtonian fluids [10, 13, 14].

The non-linear behavior of non-Newtonian fluids with respect to shear stress and deformation rate, as well as their disregard for the rule of viscosity, Put otherwise, Newtonian fluids have a constant viscosity independent of the shear tension applied which make them far more important in today's fluid mechanics than in-viscid and Newtonian fluids [1, 5].

The demand from many sectors, including the Petroleum, Chemical, Iron, and Bio Rheological fields where molten metal is taken into consideration has led to a notable increase in interest in non-Newtonian fluids in particular [17, 20]. One particular subclass of non-Newtonian fluids known as Bingham plastic fluids is distinguished by its yield stress-producing behavior. A certain minimum shear stress is necessary for the Bingham fluid to begin to flow. The fluid exhibits solid behavior and doesn't flow below this yield stress [4, 6].The fluid exhibits Newtonian behavior after the yield stress is surpassed; subsequent increases in shear stress have no effect on the fluid's viscosity[9, 19]. In this geometry of drainage of the Bingham fluid is due gravity which causes it to flow via a circular tube. The flow of the fluid due to gravity in tank produces Non-linear partial differential equations which are mainly solved by different methods [8, 18, 21].

Perturbation method is used for third order fluid [2, 12, 15], Separation of variable is used for Newtonian fluid in [20] and analytic technique is used for electrically conducting Newtonian fluid in [7, 11] exact solution of Couple stress fluid [16]. In this study emerging partial differential equation is being solved with separable technique, exact solution in terms of velocity profile for Bingham plastic fluid is obtained with the help of no-slip boundary condition. Claude-Louis Navier first presented the idea of the "no-slip boundary condition" in fluid dynamics at the beginning of the 1900s [3, 7]. The no-slip boundary condition asserts that at a solid boundary, the fluid velocity is zero. This indicates that the fluid adheres to the solid surface and does not slide beyond it. Due to typical behavior of Bingham fluid the velocity in interior region and outer region both are obtained. Further the flow rate, exerting shear stress on the pipe, average velocity mathematical relation between time and depth of tank is obtained.Then bingham fluid is reduced to the classical model when $T_0 \to 0$ and $-\mu \to \mu$ are substitute.

2 Governing equations

Neglecting the thermal effects the equation for viscous flow of incompressible fluid is defined as:

$$\nabla . V = 0. \tag{1}$$

The Equation (1) defines that the gradient of velocity is 0. In addition, it explains that flow of the fluid is not compressible which gives knowledge that volume of the fluid is constant throughout the region.

$$\rho . b + \nabla . T - \nabla p = \rho \frac{DV}{Dt}. \tag{2}$$

In first term from left side density is multiplied to body force and added to gradient of stress tensor and at the right side density is multiplied to total derivative which is equal to the convective and local derivative. [16].

3 Problem Formulation

An unsteady cylinderical tank drainnage is palced at height and flow of the fluid is considered through a pipe of Radius of R in the direction of the gravity. R_T is the Radius of tank drainage and the distance from top to bottom of tank is taken as H(t). As from geometry its is seen that H_o is the total deepness of tank.

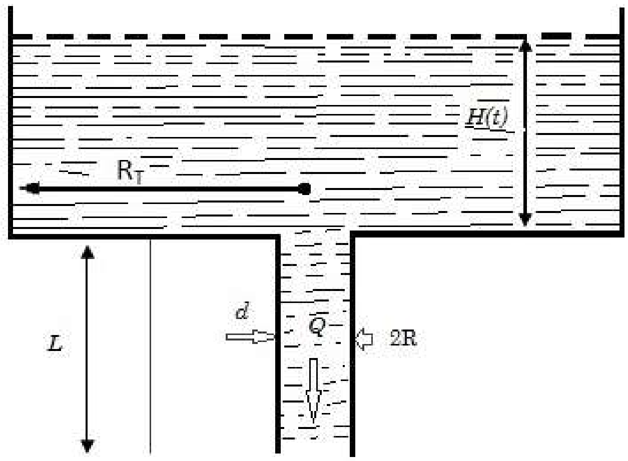

Figure 1. Tank drainage flow down via a round pipe [12, 15]

The velocity distribution, stress on the pipe of tank for Bingham plastic fluid is planned to find. So that here cylindrical co-ordinates are used in which the radial and tangential co-ordinates are zero as defined in equation (3) due to flow is considered only in axial direction for this model of tank. After then, the fluid flow is exclusively in the z-axis direction. Therefore velocity field and shear stress can be assumed as follows in equation in (3):

$$V = [v_r, v_\theta, v_z] = [\ 0, 0, V_z(r,t)\ ], \quad T = T(r,t). \tag{3}$$

Using this assumption the continuity and equation of motion is identically satisfied so that the results is as follows defined in equation (4).

$$\rho \frac{\partial u(r,t)}{\partial t} = \rho g - \frac{\partial \rho}{\partial z} + \frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r}. \tag{4}$$

It is noted that the inlet and outlet pressures are independent of each other such that the total change in pressure is the weight force divided by the length of pipe L as defined in equation (5)

$$\frac{\partial p}{\partial z} = -\frac{\rho g H(t)}{L}. \tag{5}$$

The flow of the fluid from pipe drains very slow due high yield stress of the Bingham plastic fluid so that the velocity distribution of the fluid is near to constant, due this phenomena the time derivative of the fluid vanishes and later combining and rearranging equating (4 and 5) yields results as defined as follows in equation (6).

$$\frac{\partial (rT_{rz})}{\partial r} = -\rho g r \left(\frac{H(t)}{L} + 1\right). \tag{6}$$

This equation further reduces as follows after integrating it with r component

$$rT_{rz} = -\frac{\rho g r^2}{2}\left(\frac{H(t)}{L} + 1\right) + f_1(t). \tag{7}$$

Equation (7) provides such an efficient result which includes the arbitrary function of integration in terms of t. Now in order to further solve this we are imposing the center-line and No-Slip Conditions which are suitable for this flow of the model and defined as follows respectively.

$$At \ \ r = 0, \quad T_{rz} = 0 \ \ and \ \ r = R, \quad v_z = 0 \quad . \tag{8}$$

From this it is concluded that the arbitrary function of equation of (7) is zero and further results are simplified as in equation (9)

$$T_{rz} = -\frac{\rho g r}{2}\left(\frac{H(t)}{L} + 1\right). \tag{9}$$

Here it is necessary to defined the model of Bingham plastic fluid in mathematical formulation so that is defined as that the viscoelastic fluids has no flow until and unless the external force is not applied in such that the Bingham also possess the yield stress which define the model as in equation (10 and 11)

$$\eta \to \infty \quad or \quad \frac{\partial v_z}{\partial r} = 0 \ \ if \quad |T_{rz}| \leq T_0 \, , \tag{10}$$

$$\eta = \mu_o + \frac{T_o}{\pm\frac{\partial v_z}{\partial r}} \quad or \ \ T_{rz} = -\mu_o \frac{\partial v_z}{\partial r} \pm T_0 \quad if \quad |T_{rz}| \geqq T_o \, , \tag{11}$$

In Equation (10 and 11) the terminologies are defined that η is considered as viscosity of the fluid further μ_o as the parameter of Bingham fluid. From equation (10 and 11) it is observed that the both negative and positive signs are used with T_o and change of velocity when the tensor T_rz is negative then assume that $T_{rz} = T_0$ and r is equal to the radius of plug flow $r = r_o$. Therefore the radius of flug flow from equation (9) is defined as:

$$T_o = -\frac{\rho g r_o}{2}\left(\frac{H(t)}{L} + 1\right). \tag{12}$$

Let's assume that the inner region ($r \leq r_o$) and also that the yield stress is less than or equal than the stress ($T_{rz} \leq T_o$) and the stress is defined by $T_{rz} = -\eta\frac{\partial v_z}{\partial r}$ equation (12) provide result as defined in equation (13)

$$\frac{\partial v_{zi}}{\partial r} = 0. \tag{13}$$

Integrating equation (13) with radial component we get an arbitrary function in terms of t as well

$$V_{zi} = f_2(t). \tag{14}$$

This arbitrary function defines the flug flow in inner region. The subscript (i) represents inner region and now subscript o is defining here the outer region so that for exterior region we have $r_o \leq r \leq R$,means the velocity distribution decreases when r increases so that we defined as $\frac{\partial v_z}{\partial r} \leq 0$ and $T_{rz} \geqq T_o$.Therefor the Bingham model according to the equation (11) is defined as in the equation (15).

$$T_{rz} = -\mu_o \frac{\partial v_{zo}}{\partial r} + T_o \quad or \quad \eta = \mu_o + \frac{T_o}{-\frac{\partial v_{zo}}{\partial r}} \quad for \qquad r_o \leq r \leq R. \tag{15}$$

In order to obtain the velocity distribution for $r_o \leq r \leq R$ equations (9) and (15) here T_{rz} may be eliminate to obtain the equation (16).

$$\frac{\partial v_{zo}}{\partial r} = \frac{\rho g r}{2\mu_o}\left(\frac{H(t)}{L} + 1\right) + \frac{T_o}{\mu_o} \quad for \ \ r_o \leq r \leq R. \tag{16}$$

Now after integrating the equation (16) with radial component we obtain

$$v_{zo} = \frac{\rho g r^2}{4\mu_o}\left(\frac{H(t)}{L}+1\right) + \frac{T_o r}{\mu_o} + f_3(t)\,, \tag{17}$$

Where $f_2(t)$ is the arbitrary function so that utilizing the boundary condition $v_{zo} = 0 \quad r = R$. so that the velocity distribution in the outer region is obtain as in

$$v_{zo} = -\frac{\rho g r^2}{4\mu_o}\left(\frac{H(t)}{L}+1\right)\left\{1-\left(\frac{r}{R}\right)^2\right\} - \frac{T_o R}{\mu_o}\left\{1-\frac{r}{R}\right\}. \tag{18}$$

To conclude $f_2(t)$, it possibly will be noted that $v_{zi} = v_{zo}$ at $r = r_o$.Then equation (18) along with (10) may be shortened to obtain,

$$f_2(t) = -\frac{\rho g r^2}{4\mu_o}\left(\frac{H(t)}{L}+1\right)\left(1-\left(\frac{r_o}{R}\right)^2\right). \tag{19}$$

Therefore the velocity distribution in the plug flow in the inner region is obtained as follows.

$$v_{zi} = -\frac{\rho g r^2}{4\mu_o}\left(\frac{H(t)}{L}+1\right)\left(1-\left(\frac{r_o}{R}\right)^2\right) \quad for \;\; r \leq r_o\,. \tag{20}$$

here $r_o = \frac{-2T_o L}{\rho g(H(t)+L)}$ represents the radius of the plug-flow zone at the pipe's center. The velocity profile is parabolic in the outer area (equation 20), and flat in the inner region (equation 18).

Flow rate

The "flow rate" may be calculated by integrating the velocity distribution over the cross section of the circular pipe, as described in [2]. Rather of entering two independent velocity formulas from (18) and (20) and integrating in two areas, integrating by parts yields the required result.

$$Q = 2\pi\left[\frac{v_z r^2}{2}\Big|_0^R - \frac{1}{2}\int_0^R r^2\left(\frac{\partial v_z}{\partial r}\right)dr\right]. \tag{21}$$

The first term in the square brackets above is zero at both limits on using no-slip boundary condition at the upper limit. From equation (9), $\frac{r}{R} = \frac{T_{rz}}{T_R}$ where $T_R = -\frac{\rho g R}{2}\left(\frac{H(t)}{L}+1\right)$ is the wall shear stress. Thus, a general expression for the flow rate in a circular pipe is

$$Q = \frac{\pi R^3}{T_R^3}\int_0^{T_R} T_{rz}^2\left(-\frac{\partial v_z}{\partial r}\right)dT_{rz}. \tag{22}$$

The lower limit of integration is reset to T_o using equation (10). Then, substitute equation (15) and integrate produces

$$Q = \frac{\pi R^3}{\mu_o T_R^3}\int_{T_o}^{T_R} T_{rz}^2\left(T_{rz}-T_o\right)dT_{rz} = \frac{\pi R^3 T_R}{4\mu_o}\left[1-\frac{4}{3}\left(\frac{T_o}{T_R}\right)+\frac{1}{3}\left(\frac{T_o}{T_R}\right)^4\right]. \tag{23}$$

By replacing $T_0 \to 0$ and $-\mu \to \mu$, the Bingham model reduces to the Newtonian model.

Average Velocity

We determine the average velocity by using the formula defined in [2] so the average velocity of the fluid flowing down the pipe is obtained as

$$\vec{V} = \frac{R T_R}{4\mu_o}\left[1-\frac{4}{3}\left(\frac{T_o}{T_R}\right)+\frac{1}{3}\left(\frac{T_o}{T_R}\right)^4\right]. \tag{24}$$

Shear Stress on the pipe

Shear stress on the pipe is given by

$$T_{rz}\Big|_{r=R} = -\frac{\rho g R}{2}\left[\frac{H(t)}{L} + 1\right]. \tag{25}$$

Mass balance over the entire tank

Mass balance throughout the whole tank is described in [2], putting flow rate from equation (24) into the mass balance formula established in [2],and then separating variables on both sides of the equation, one obtains

$$\mathrm{H(t)} = (H_O + L) \times e^{\frac{\rho g R^4 t}{8\mu_O L R_T^2}\left[1 - \frac{4}{3}\left(\frac{r_o}{R}\right) + \frac{1}{3}\left(\frac{r_o}{R}\right)^4\right]} - L. \tag{26}$$

Relationship between time and depth of fluid in the tank

The relationship between time and the depth of the fluid present in the tank can be separately written as

$$\mathrm{t} = \frac{8\mu_O L R_T^2}{\rho g R^4\left[1 - \frac{4}{3}\left(\frac{r_o}{R}\right) + \frac{1}{3}\left(\frac{r_o}{R}\right)^4\right]} ln\left(\frac{H(t) + L}{H_O + L}\right). \tag{27}$$

Time of efflux

The time required for complete drainage (Time of efflux) is obtained by taking $H(t) = 0$

$$\mathrm{t}_{eff} = \frac{8\mu_O L R_T^2}{\rho g R^4\left[1 - \frac{4}{3}\left(\frac{r_o}{R}\right) + \frac{1}{3}\left(\frac{r_o}{R}\right)^4\right]} ln\left(\frac{L}{H_O + L}\right). \tag{28}$$

4 Results and Discussion

In the preceding sections, we investigated the unsteady tank drainage problem using an isothermal, incompressible Bingham plastic fluid with a no-slip condition. We have obtained an exact solution for the nonlinear partial differential equation. Variations in velocity profile v_z and depth $H(t)$ with the influence of different parameters, including density ρ, pipe radius R, Bingham model parameter μ_O, pipe length L, tank depth, and the radius of the plug flow region R_o are being examined, as shown in Figures. 2-6 . Also the effect of tank depth with respect to time is described as in figures 7-11. In Figures 2-6 It is being observed that increasing the no-slip parameter, plug flow region, Bingham model parameter, and pipe length lead to decrease in velocity. Conversely, increasing the parameters of density and pipe radius resulted in an increase in velocity. In Figures 7-11 it is being found that increasing the plug flow region radius, Bingham model parameter, and pipe length lead to an increase in the depth of the tank with respect to time. On the other hand, increasing the parameter of density and pipe radius lead to a decrease in the depth of the tank with respect to time.

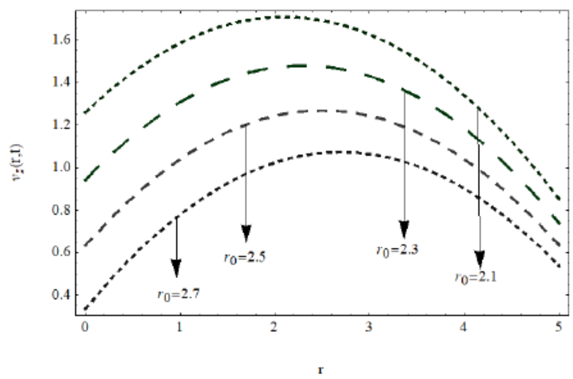

Figure 2. Effect of r_o on velocity field, when ρ = 1.38g/cm^3, μ_o = 100cP, R = 5cm, L = 10cm, $H(t)$ = 20cm

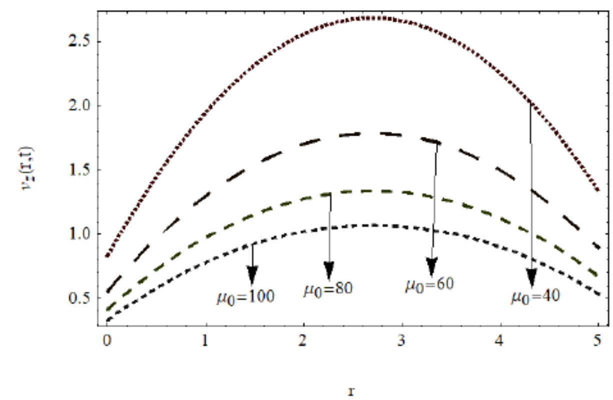

Figure 3. Influence of μ_o on velocity distribution, when R = 5cm; ρ = 1.38g/cm^3, L = 10cm; $H(t)$ = 20cm

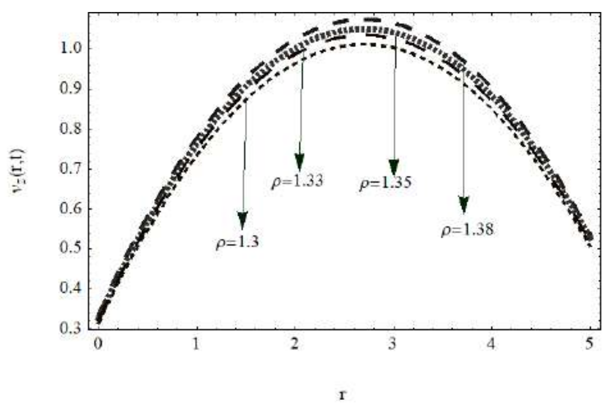

Figure 4. Effect of ρ on velocity profile, when μ_o = 100cP, R = 5cm, L = 10cm, $H(t)$ = 20cm, r_o2.7 = cm

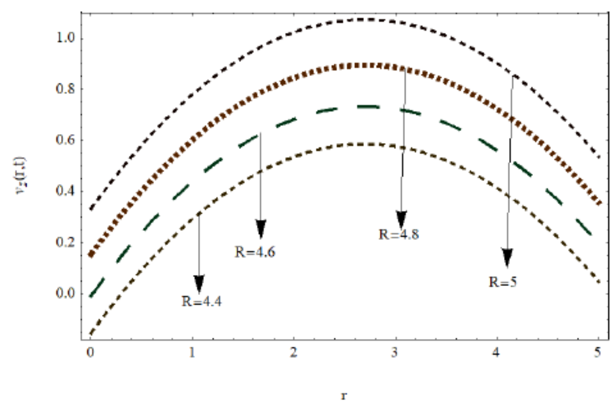

Figure 5. Effect of R on velocity profile, when ρ1.38g/cm^3 μ_o = 100cP; L = 10cm; $H(t)$ = 20cmr_o = 2.7cm

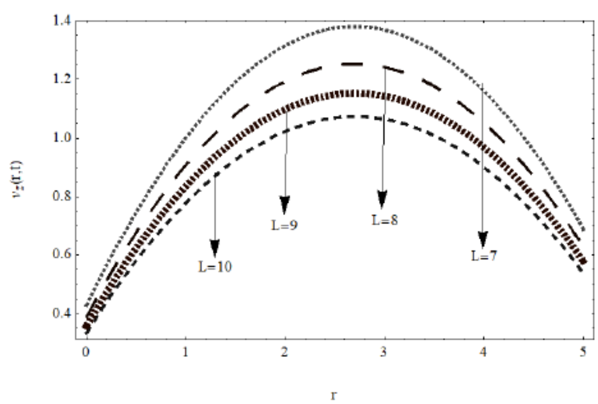

Figure 6. Effect of L on velocity profile, when ρ = 1.38g/cm^3, μ = 100cP, R = 5cm, $H(t)$ = 20cm, r_o = 2.7cm

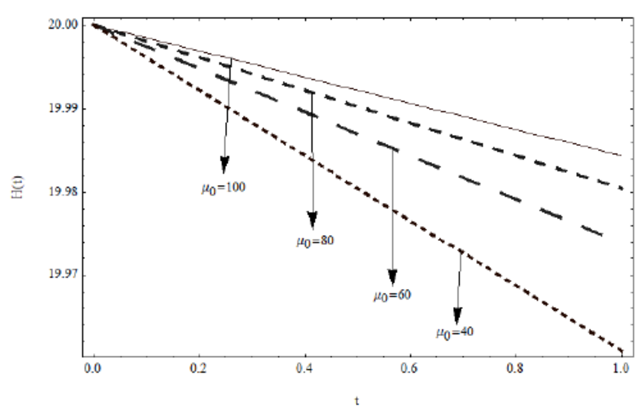

Figure 7. Effect of μ_0 on depth w.r.t t, when ρ = 1.38g/cm^3, L = 10cm, R = 5cm, $H(t)$ = 20cmr_o = 2.7cm

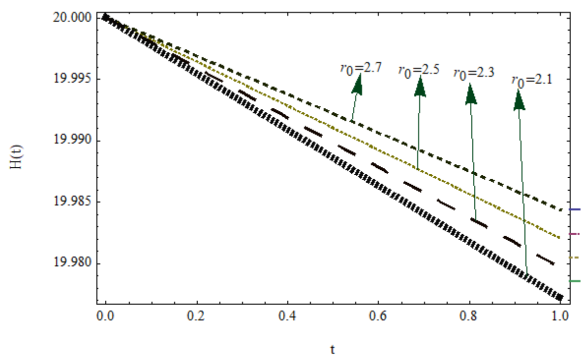

Figure 8. Effect of r_o on depth w.r.t t, when $\rho = 1.38g/cm^3, L = 10cm, R = 5cm, H(t) = 20cm\mu = 100cP$

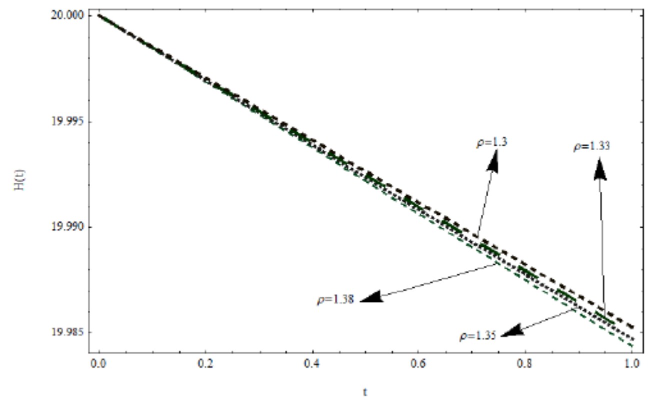

Figure 9. Effect of ρ on depth w.r.t t, when $R = 5cm; \mu = 100cp; L = 10cm; H(t) = 20cm, r_o = 2.7cm$

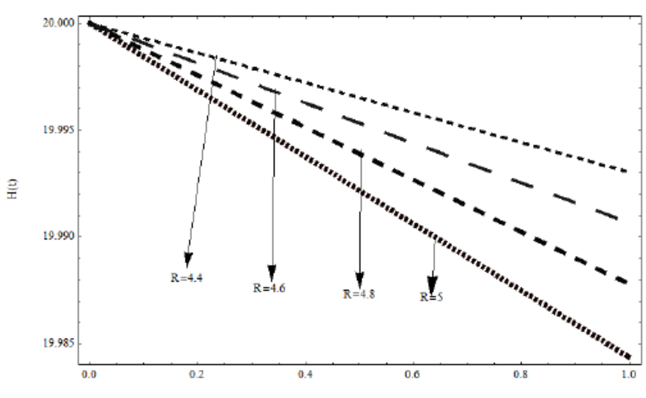

Figure 10. Effect of R on depth w.r.t t, when $\rho = 0.78g/cm^3, \mu = 11.5cP, textcm, L = 10cm, H(t) = 20cm, r_o = 2.7cm$

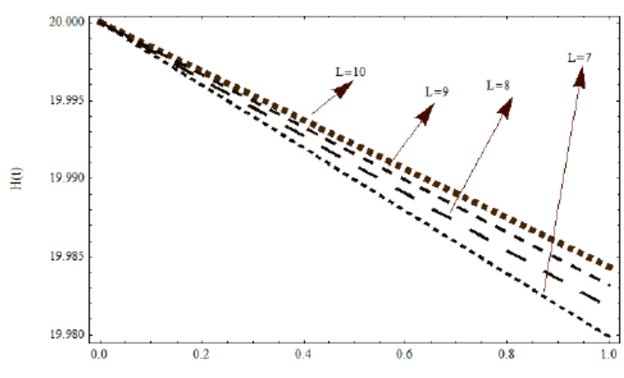

Figure 11. Effect of L on depth w.r.t t, when $R = 5cm; \mu = 11.5cP; H(t) = 20cm, r_o = 2.7cm$

5 Conclusion

In this study unsteady, incompressible, an isothermal tank drainage model for Bingham plastic fluid with no-slip condition is being studied. The exact solution of emerging Non linear Partial differential equation of model in terms of velocity and shear stress is obtained using separation of variables method. Furthermore flow rate, time efflux, mathematical relation between time and depth of tank is obtained. It has been observed that when the Bingham model parameters, flow region and pipe length increase, the time efflux of tank also increases.In addition, our findings indicate that higher density and a larger pipe radius result in quicker drainage of Bingham plastic fluid from the circular tank. Velocity profile of the Bingham plastic fluid in this model increases when the dynsity, radius of plug flow region is increasing. Also it is concluded that greater the yield stress greater force is required to drain the Bingham plastic fluid.

6 Author Contributions

Ghulam Murtaza: Methodology, Original draft preparation **sayed feroz shah**:Supervision. **K.N Memon** : Writing- Reviewing. **Azam Ali Amur**: Software and Editing.

7 Compliance with Ethical Standards

It is declare that all authors don't have any conflict of interest.

8 Funding Information

No Funding

9 Author Information

ORCID:
Ghulam Murtaza Oghai:: 0009-0006-7611-7194
Syed Feroz Shah:: 0000-0002-6964-1469
K.N. Memon:: 0000-0003-2163-2552
Azam Ali Amur:: 0009-0005-1746-6290

References

[1] Alam, M. K., Memon, K., Siddiqui, A., Shah, S., Farooq, M., Ayaz, M., Nofal, T. A. and Ahmad, H. [2021], 'Modeling and analysis of high shear viscoelastic ellis thin liquid film phenomena', *Physica Scripta* **96**(5), 055201.

[2] Amur, A. A., Memon, K., Shah, S. F., Amur, M., Ahmad, H., Siddiqui, A., Ozsahin, D. U. and Askar, S. [2023], 'The hydrodyanamics of gravity-driven vessel drainage of third order fluid using perturbation method', *Heliyon* **9**(10).

[3] Amur, M. A., bux Amur, K., Amur, A. A., Amur, I. A. and Memon, K. [2023], 'Adaptive numerical regularization for variational denoising model with complementary approach', *VFAST Transactions on Mathematics* **11**(2), 77–92.

[4] Dang, J., Duan, X. and Tian, S. [2022], 'Wall effects for spheroidal particle in confined bingham plastic fluids', *ACS omega* **7**(43), 38717–38727.

[5] Forbes, L. K. and Hocking, G. C. [2010], 'Unsteady draining of a fluid from a circular tank', *Applied mathematical modelling* **34**(12), 3958–3975.

[6] Fusi, L., Housiadas, K. D. and Georgiou, G. C. [2020], 'Flow of a bingham fluid in a pipe of variable radius', *Journal of Non-Newtonian Fluid Mechanics* **285**, 104393.

[7] Garg, V. K. [1998], *Applied computational fluid dynamics*, CRC Press.

[8] Hayat, T., Ahmed, N. and Sajid, M. [2008], 'Analytic solution for mhd flow of a third order fluid in a porous channel', *Journal of Computational and Applied Mathematics* **214**(2), 572–582.

[9] Hayat, T. and Kara, A. [2006], 'Couette flow of a third-grade fluid with variable magnetic field', *Mathematical and Computer Modelling* **43**(1-2), 132–137.

[10] Khaskheli, M. A., Memon, K. N., Sheikh, A. H., Siddiqui, A. M. and Shah, S. F. [2020*a*], 'Tank drainage for an electrically conducting newtonian fluid with the use of the bessel function', *Eng. Technol. Appl. Sci. Res* **10**(2).

[11] Khaskheli, M. A., Memon, K. N., Sheikh, A. H., Siddiqui, A. M. and Shah, S. F. [2020*b*], 'Tank drainage for an electrically conducting newtonian fluid with the use of the bessel function', *Eng. Technol. Appl. Sci. Res* **10**(2).

[12] Mahar, M. A., Memon, K. N., Shah, S. F., Amur, A. A. and Siddiqui, A. [2022], 'Effect of slip condition on unsteady tank drainage flow of third order fluid'.

[13] Memon, K., Khan, S. A., Islam, S., Zafar, N. A., Shah, S. F. and Siddiqui, A. [2014], 'Unsteady drainage of electrically conducting power law fluid', *Applied Mathematics & Information Sciences* **8**(5), 2287.

[14] Memon, K. N., Alam, M. K., Baili, J., Nawaz, Z., Shiekh, A. H. and Ahmad, H. [2021], 'Analytical solution of tank drainage flow for electrically conducting newtonian fluid', *Thermal Science* **25**(Spec. issue 2), 433–439.

[15] Memon, K., Shah, S. and Siddiqui, A. [2018], 'Exact solution of unsteady tank drainage for ellis fluid', *Journal of Applied Fluid Mechanics* **11**(6), 1629–1636.

[16] Memon, K., Siddiqui, A. and Shah, S. F. [2017], 'Exact solution of tank drainage through the circular pipe for couple stress fluid', *J. Appl. Environ. Biol. Sci* **7**(12), 27–34.

[17] Rother, M. A. [2023], 'Modelling tank drainage using a simple apparatus', *International Journal of Mathematical Education in Science and Technology* pp. 1–13.

[18] Ryltseva, K., Borzenko, E. and Shrager, G. [2020], 'Non-newtonian fluid flow through a sudden pipe contraction under non-isothermal conditions', *Journal of Non-Newtonian Fluid Mechanics* **286**, 104445.

[19] Shah, S. A. R., Memon, K., Shah, S., Sheikh, A. and Siddiqui, A. [2022], 'Delta perturbation method for thin film flow of a third grade fluid on a vertical moving belt', *STATISTICS, COMPUTING AND INTERDISCIPLINARY RESEARCH* **4**(1), 61–73.

[20] Shah, S. M., Memon, K., Shah, S. F., Sheikh, A. H., Ghoto, A. A. and Siddiqui, A. [2019], 'Exact solution for ptt fluid on a vertical moving belt for lift with slip condition', *Indian Journal of Science and Technology* **12**, 30.

[21] Zadorozhnyi, A., Chovnyuk, Y., Stakhovsky, O., Kovrevski, A. and Buhaievskyi, S. [2023], Modeling the flow of a bingham plastic fluid through a circular pipeline with different wall properties, *in* 'AIP Conference Proceedings', Vol. 2490, AIP Publishing.