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Order Structured Graphs of Cyclic Groups and Their Classification

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Abstract

Let $\Gamma^o(G)$ with $G \cong C_p$, a cyclic group of order p, be an order structured graph. The group C_p will be assumed as the vertex set of the graph $\Gamma^o(G)$ and an edge between vertices will be built on the basis of a defined relation via order structure. Certain graphical parameters such as independence number, clique number, domination number, and separability are discussed. Some characterizations are proposed and proved by incorporating the defined relation. It is further proved that $\Gamma^o(C_p)$ can never be a hamiltonian graph. Lastly, It is shown that $C(\Gamma^o(C_p))$ is isomorphic to $\Gamma^o(C_p)$.

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1 Introduction

Algebra, a branch of mathematics, deals with the manipulation of the symbols and the investigation of certain binary operations. It facilitates in the representation of relationships via symbols. Groups are significantly used in various contexts like chemistry, sociology, physics, biology, mathematics and technology. In chemistry elements are grouped on the basis of like features. Groups are beneficial to understand the chemical nature of compounds. Groups also play a crucial role in social structures and their relationships. Certain species are categorized according to their evolutionary relations. In information technology codes are commonly organized into certain classes for better arrangements. Groups serve as a basic concept in abstract algebra. Cyclic groups that are generated by a single element have dynamic applications in many practical contexts.

Cyclic groups are used to analyze digital signal processing specifically in the depiction of periodic signals. In the study of finite state machines to comprehend the algorithmic processes cyclic groups are employed.

These structures help to study the certain periodic behaviors. Cyclic groups play a vital role to understand the properties that are relevant to divisibility and congruences. Also the cyclic codes that are derived from cyclic groups are used in coding theory for error detection. Prime order cyclic groups provide a basis to understand the features of prime numbers and their relations with congruence theory.

Group theory is crucial in coding theory. The prime order cyclic groups perform a strong connection among various disciplines. These groups are of primary importance to comprehend the properties of groups. Prime order cyclic groups are interlinked to number theory especially to modular arithmetic.

Many cryptographic protocols are constructed with enhanced security of cryptographic systems with the aid of the prime order groups generated by a single element only. Moreover, groups are fundamental for end to end encrypted network communications in present digital era.

The importance of cyclic groups depends upon their ability to sketch periodic behaviors in numerous mathematical disciplines. Cyclic groups have a number of applications in theoretical and practical domains. Groups have dynamic application in mathematics and beyond. They supply a framework to comprehend transformations and symmetries. Groups have applications in graphics, cryptography and computer science too.

The graphs associated with groups and the set of generators for that group were investigated in 2008 [13]. Some generalized and basic concepts of algebra with a variety of examples are explained in [11]. Bertram [3]employed graph theory practically to solve certain daily life problems. Bertrum and et. al explored and investigated many interesting connections between the parameter of graphs and the conjugacy classes of groups. Burton [4] expressed a variety of relations between certain positive integers that also had a wide combinatorial approach.

Certain practical applications of combinatorial graph theory have been elabortaed in [2] A. Erfanian and B. Tolue [8] proposed and validated the conjugate graphs for non-abelain group G, denoted as Γ_G^c. All the vertices except those of lying in center worked as vertex set. Two conjugate vertices are adjacent. Both established a relationship between chromatic numbers and clique number. The non-hamiltonain properties are also investigated for these special graphs.

The spectral characteristics of different types of graphs are investigated in [6, 10, 14]. The energy of these graphs is also discussed in these articles.

Laplacian energy, a special kind of energy is also computed. R. Rajkumar and P. Devi [15] introduced and analyzed the graphs constructed on the cyclic subgroups. Where two cyclic subgroups are linked via an edge if both are permutable. Peter J. Cameron also made an addition to algebraic based structured graphs, where an edge did exist under some specific group structure.

He investigated the enhanced power graphs, an extension of power graphs. The isomorphism relation for power graphs was also produced. The maximal cyclic subgroups and maximal clique were made linked to each other. S. Ali [16] and et. al investigated the new class of integers like totient integers. The number 2^α where $\alpha \neq 0$ and $\alpha = 2t, t \geq 1$ was proved to be an anti-totient number. Also various spectral properties of these special integers were discussed. D. A. Juraev and M. N. Bozorov [12] investigated the significance of algebra as well as its applications in some practical issues. Algebra in instrumental wile dealing with the variables. That is highly useful in data analysis.

A.Alameri [1] and et. al worked on some special patterns observing connectivity among the vertices of various families of graphs.

Neighbourhoods and degree based investigations has applications in science, physics and social network analysis. G. D. I. Luna [7] and et. al illustrated some counting rules for the computation of independent sets that has applications in combinatorics.

Mini Gopalkrishnan [9] and et. al explored a special algebraic structured graph having embedding of K_3 graphs. They named these graphs as cyclic graphs. It was established that the enumeration of K_3 graphs is interlinked with the number of non-generators and the number of generators. Further they presented the practical approach of algebra in several genetical phenomenon. That emphasized on the significance of algebra in our daily lives. S. Chattopadhyay and P. Panigrahi [5] worked on the classification of periodic cyclic and dicyclic groups. Both investigated a new family of graph known as power graphs. That contains the elements of the group as the vertex set. The adjacency relation between any two vertices exists only if one is expressed as an integral power of other.

Further he validated these power graphs under non-hamiltonain phenomenon. Next we introduce a novel structured graphs built on the finite order cyclic groups. Before that algebraic structured graphs were investigated considering the adjacency relation between the elements having order as power of the order of the other element. The newly constructed graphs are termed as order structured simple graphs (OSSGs).

Only the distinct order vertices are made adjacent and the graph degree sum rely on the order and thus the degree of the vertices is also discussed. Further we have classified the graphs by using the concept of connectivity among its vertices.

2 Preliminaries And Basic Definitions:

Initially we elaborate some of the main concepts from abstract algebra and graph theory. In a graph, degree of a vertex v_k is the total number of edges that are linked with v_k. If any vertex has even degree then the vertex is called even otherwise odd. The sum of all degrees of all vertices that are present in the graph is called the whole graph degree.

A tree is a special subgraph having order same as that of the graph with size one less than the order. A connected subgraph in which each vertex is reachable from any vertex of the connected subgraph is called a component. A cut vertex is a vertex whose removal from the graph increases the number of components of the graph. If a graph has a single cut vertex then the graph becomes hamiltonian and separable. A dominating set contains all those vertices that are always connected to all the remaining vertices in that graph. The radius of a graph has all those number of vertices whose eccentricity is minimum.

The clique number of a graph is defined as the total number of edges in its largest complete subgraph. It's just like to find the maximum group of friends in the graph in which everyone is a friend of one other. A square matrix defined as the difference between the degree and the adjacency matrices of a graph is termed as a Laplacian matrix.

It assists in to analyze connectivity and graph cuts. In a connected graph the number of spanning trees can be computed by taking determinant of any cofactor of Laplacian matrix. The Fiedler eigenvalue is the second smallest eigenvalue of the special matrix that is Laplacian matrix of that graph.

Graph partitioning as well as community detection algorithms rely on the notion. It also helps to understand the structure of the graph. The Estrada index was first introduced by Ernesto Estrada, a mathematician and physicist, in the field of network science. A numerical value calculated from the exponentiation of the adjacency spectrum of the graph and summing up over all of its eigenvalues. It explains the that how well connected the vertices are.

The largest subset of non-adjacent vertices is termed as an independent set. The set of all adjacent vertices to any specific vertex is titled as an open neighborhood.If the open neighborhood includes that specific vertex too then we get a closed neighborhood.

3 Exploring The Graphical Parameters For The Order Structured Simple Graph(OSSG):

Some of the key concepts regarding algebra and graphs is presented in sections 1 and 2. Next we begin section 3 by defining the concept of the order structured simple graphs empowered by the related figures.

The relation between any arbitrary vertex degree and the degree of the identity element is developed. We will also compute many graphical measures like separability, independence number, cut vertices, neighbourhoods, energy, Ernesto Estrada index etc. The clique number for $\Gamma^o(C_p)$ is always fixed that remain invariant for any choice of p. The degree of any vertex of the cyclic group and Euler Phi function is interlinked.

In this section, graph degree sum for $\Gamma^o(C_p)$ with p as a prime, is observed. In later work, a general criterion for counting of even and odd vertices is also explored. We discuss some general results relevant to the open and closed neighborhoods of each element of the vertex set. Now let's begin the section via main definition stated as,

Definition 1. *Let $\Gamma^o(C_p)$ be a simple graph defined on the basis of the order structure of elements of C_p. The vertex set of the graph has all the elements of of C_p, usually denoted by $V(\Gamma^o(C_p))$ or by C_p. Any two distinct vertices u and v of $\Gamma^o(C_p)$ are adjacent if both the vertices differ in orders.*

The basic concept of the definition is illustrated with the help of following examples and the graphs as expressed in FIG. 1.

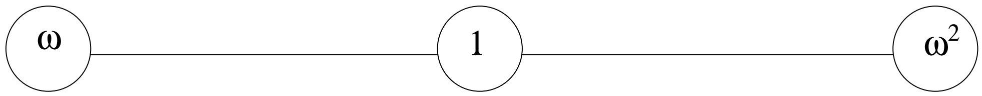
If $p = 3$ in C_p then the OSSG constructing for C_p contains all the elements of $C_3 = \{1, w, w^2\}$ as its vertices. As 1 is only element with order 1. So it remains adjacent to w and w^2. As $w^{-1} = w^2$ and $(w^2)^{-1} = w$ that's why $w \nsim w^2$. So we get the graph $\Gamma^o(C_3)$ as displayed in FIG. 1, that is isomorphic to P_3(a path graph of order 3).

If $p = 4$ then we have C_4 as the four forth roots of unity that is $C_4 = \{\pm 1, \pm i\}$, which produces a cyclic group generated by i. The order of its elements is as, $o(1) = 1, o(-1) = 2, o(i) = 4, o(-i) = 4$. Here 1 and -1 have orders that differ from orders of all other elements. Therefore, $1 \sim -1, 1 \sim i, 1 \sim -i, -1 \sim i, -1 \sim -i$. But i and $-i$ are inverses of each other thus by the definition of OSSG these two elements are not adjacent as shown in FIG. 1.

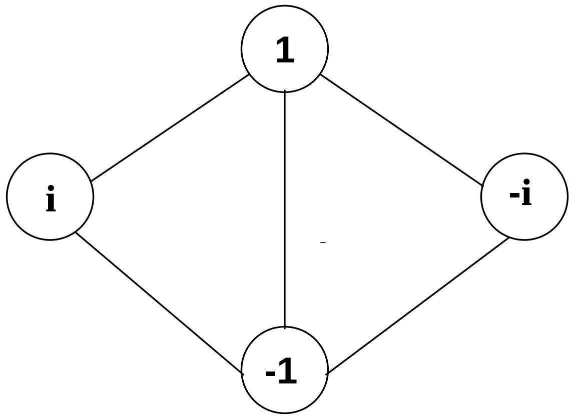
The following result illustrates that the degree of the element e is always greater than or equal to the degree of any arbitrary vertex of the group generated by a single element.

Theorem 1. *In $\Gamma^o(C_p)$, $deg(a^i) \leq deg(e)$ where a^i denotes an arbitrary non-identity element of the cyclic group and e is the identity element of C_p.*

Proof. For C_p the corresponding order structured graph $\Gamma^o(C_p)$ built a star graph. And $deg(e) = p-1$. Further let a^i be any non-identity vertex of $\Gamma^o(C_p)$. Then to determine its degree firstly we define the total number of vertices having same order as that of a^i. Let $o(a^i) = d_i$ where $d_i \neq 1$ with d_i divides p. Then evidently $deg(a^i) = p - \phi(d_i)$. Also as $d_i \neq 1$ then the only possibility is that $d_i = p$. Also for the general $\phi(d_i) \geq 1$. Hence $p - \phi(d_i) < p - 1 >$ It further concludes that the degree of e is always less than from the degree of any non-identity vertex. □



$[\Gamma^o(C_3)]$



$[\Gamma^o(C_4)]$

Figure 1. The graphs in (*a*) and (*b*) both represent the relation between the elements of the group and their orders. It is discovered that (*a*) represents a path graph but the graph in (*b*) is not a path graph.

The below result specifies the number of the spanning trees present in $\Gamma^o(C_p)$.

Theorem 2. $\Gamma^o(C_p)$ *has a single subgraph constructing a spanning tree of the order structured graph for* C_p.

Proof. For the cyclic group C_p, the related order structured graph $\Gamma^o(C_p)$ has order p. And only e is the vertex of unique order and all the remaining $p-1$ vertices are definitely of order p. That is $p-1$ vertices are not linked with each other but these vertices are adjacent to e only. Therefore it produces a single spanning tree with order p and size $p-1$. □

For the order structured simple graph generated for a cyclic group, we discuss the degree of each vertex of the graph and then ultimately we add up all these degrees to produce the whole graph degree.

Theorem 3. *If* $G \cong C_p$ *in* $\Gamma^o(G)$*, where G is a group of prime order then* $\sum deg(\Gamma^o(G))$ *is* $2(p-1)$.

Proof. Let $G \cong C_p$ be a finite group generated by a single element. And $\Gamma^o(G)$ represents the order structured graph of prime order cyclic group. Then surely, G has p vertices. Next, we will observe the degree of vertices, as it constitutes the whole graph degree. Also $e \in C_p$ with $o(e) = 1$ and for arbitrary $a^i \in C_p$ with $1 \leq i < p$, $\gcd(i, p) = 1$. Therefore, $o(a^i) = p$. That is $(p-1)$ vertices are of identical degrees. Clearly $o(e) \neq o(a^i)$ for $1 \leq i < p$. So $e \sim a^i$. It further implies that e has degree $(p-1)$. And $(p-1)$ elements remain attached to e only. Therefore, $p-1$ vertices are of degree 1. Thus the sum of all degrees is, $p-1+(p-1)$ that is $2(p-1)$ is the total graph degree of $\Gamma^o(G) \cong \Gamma^o(C_p)$ for $G \cong C_p$. □

The open neighbourhood of each of the non-identity element contains only a single element e. But for the closed neighbourhood for such element we also include that element too.

Theorem 4. *For* $\Gamma^o(C_p)$*, the open and closed neighborhood of its non identity vertices depends upon the neutral element.*

Proof. Let C_p be a prime order cyclic group. $\Gamma^o(C_p)$ is the order structured simple graph of C_p. Now firstly to evaluate open neighborhood of all non identity elements $a^i \in C_p, 1 \leq i \leq p-1$, we observe all the edges that are adjacent to a^i. Clearly C_p constitutes a star graph. As e is a vertex of the graph with degree $p-1$. Therefore, every a^i is always adjacent to e. Now we see the pattern of other edges of a^i that are produced among a^i. As $\Gamma^o(C_p)$ is a star graph with $deg(e) = p-1$ so it implies that $deg(a^i) = 1, 1 \leq i \leq p-1$. This definitely exists as each exponent of a is co-prime to p so, $O(a^i) = p$. Further the vertices a^i are not connected with each other for being of having same orders. Ultimately, only a single edge is produced for each such a^i that is with end points e and a^i. Therefore,

$$N(a^i) = \{e\}$$

Now the closed neighborhood of the vertex a^i is simply computed as,

$$N[a^i] = N(a^i)\bigcup\{a^i\}$$

So it implies that,

$$N[a^i] = \{e, a^i\}$$

□

In next result we enumerate the even and odd vertices of $\Gamma^o(C_p)$. We will further conclude that for any odd prime, degree of e is always one less than the order of this graph.

Theorem 5. *In $\Gamma^o(C_p)$ for $C_p =< a : a^p = 1 >$, only neutral element that coincides with the identity element e is even and the other are all odd vertices.*

Proof. Let $C_p =< a : a^p = 1 >$ be a group of order p. And $\Gamma^o(C_p)$ be an order structured graph of C_p then clearly $deg(e) = p-1$, where e stands for the neutral element. As p is odd so $(p-1)$ generates an even number. Therefore e is always an even degree vertex. Further, $(p-1)$ vertices are adjacent to e only, so we induce that $deg(a^i) = 1$, for $1 \leq i < p$. That is $(p-1)$ vertices are of odd degree. Ultimately $\Gamma^o(C_p)$ has one only even vertex and $(p-1)$ odd vertices. □

For the independence number, we take all those vertices that are pair-wise non-adjacent.

We consider for this the vertices of $\Gamma^o(C_p)$ which have only a single vertex in their open neighbourhood of each of these vertices. So the total number of non-identity elements of C_p are included in its largest independent set.

Lemma 1. *For $\Gamma^o(C_p)$ the independence number is $(p-1)$.*

Theorem 6. *If $\Gamma^o(C_p)$ is an order structured graph of C_p, C_p being a group generated by a single element having order p then,*

$$IR(\Gamma^o(C_p)) = \frac{p-1}{p}$$

where IR stands for independence ratio.

Proof. Let $C_p =< a : a^p = 1 >$ be a cyclic group of order p. From Lemma 4.6 it is obvious that $(p-1)$ non-trivial vertices are not inter linked because of having like orders. Therefore, the vertex independence number $\beta(\Gamma^o(C_p)$ remains $(p-1)$. Further $|C_p| = p$. Thus IR an independence ratio being a ratio of $\beta(\Gamma^o(C_p)$ and p is expressed as,

$$IR(\Gamma^o(C_p) = \frac{p-1}{p}$$

□

The result illustrates that the identity element is linked with every vertex, that's why the open neighbourhood has order one less than the order of the group.

Theorem 7. *In $\Gamma^o(G)$ with $G \cong C_p$ the open neighbourhood $N(e)$ $N(e)$ of the identity element e, contains all the non-identity elements.*

Proof. We recall that an open neighborhood of any vertex $v \in G$ contains all those vertices that are adjacent to the specific vertex v. Let us consider $e = v$. As in $\Gamma^o(C_p)$ all non- identity elements are adjacent to e, therefore all vertices with the exception to e falls in open neighborhood of e. That is,

$$N(e) = \{a^i : 1 \leq i < p\}.$$

As $i \neq 0$, so $a^0 \notin N(e)$. It implies that $1 \notin N(e)$. Alternatively, for group $G =< a : a^p = 1 >$ the open neighborhood of e can be written as,

$$N(e) = G - \{e\}.$$

□

The forthcoming result extends the notion of the closed neighbourhood. That includes the same number of vertices as that of $\Gamma^o(C_p)$.

Corollary 1. *In an order structured graph of a group generated by a single element a, having order p,*

$$N[e] = G.$$

where $N[e]$ stands for the closed neighborhood of e.

A relationship between the exponential eigen values and a special index named as Ernesto Estrada Index is established in the following result, by using the concept of a bipartite graph.

Theorem 8. *Let $\Gamma^o(C_p)$ be an order structured graph of C_p, where $C_p =< a : a^p = 1 >$ then*

$$EE(\Gamma^o(C_p)) = \frac{1}{e^{\sqrt{p-1}}}[1 + (e^{\sqrt{p-1}})^2 + p - 2].$$

where $EE(\Gamma^o(C_p))$ stands for the Ernesto Estrada index.

Proof. We know that for a complete bipartite graph with the vertex set partitioning having r and s elements. Where $r + s = p$ has adjacency spectrum as, $-\sqrt{rs}$, 0^{r+s-2}, $\sqrt{rs}$. Also $\Gamma^o(C_p)$ satisfies that $deg(e) = p - 1$. And $deg(a^i) = 1$ for $1 \leq i < p$ that is conforming to the definition of a star graph. Therefore, $\Gamma^o(C_p) \cong K_{1,p-1}$. Thus the adjacency spectrum for $r = 1$ and $s = p - 1$ reduces to $-\sqrt{p-1}$, 0^{p-2}, $\sqrt{p-1}$. In terms of eigen values λ_i for $1 \leq i \leq p$ with respect to the adjacency matrix, it can be expressed as, $\lambda_1 = -\sqrt{p-1}, \lambda_2 = \lambda_3 = \lambda_3 = \cdots = 0, \ \ \lambda_p = \sqrt{p-1}$. Also,

$$\begin{aligned}\sum_{i=1}^{p} e^{\lambda_i} &= e^{\lambda_1} + e^{\lambda_2} + \ldots + e^{\lambda_p} \\ &= e^{-\sqrt{p-1}} + (p-2) + e^{\sqrt{p-1}} \\ \sum_{i=1}^{p} e^{\lambda_i} &= \frac{1}{e^{\sqrt{p-1}}}[1 + (e^{\sqrt{p-1}})^2 + p - 2].\end{aligned}$$

□

As $\alpha(G^{\star}) + \beta(G^{\star}) = n$ for a connected graph $G^{\star}$ of order n with no isolated vertices. As $\Gamma^o(C_p)$ is a star graph of order p. So $(p-1)$ isolated vertices are present. Next we discuss the vertex covering number.

Theorem 9. *For $\Gamma^o(C_p)$, $\alpha(\Gamma^o(C_p))$ always remains constant for any choice of p.*

Proof. It is obvious that,

$$\begin{aligned}\alpha(\Gamma^o(C_p)) + \beta(\Gamma^o(C_p)) &= |C_p| \\ \alpha(\Gamma^o(C_p)) &= p - \beta(\Gamma^o(C_p))\end{aligned}$$

Now for $\beta(\Gamma^o(C_p)) = p-1$, the expression leads to produce the value of $\alpha(\Gamma^o(C_p))$ (that is same as that of the complement of the vertex independence number) is equal to one. □

Further we see the clique number of the order structured simple graph on the basis of a complete subgraphs.

Theorem 10. *Let G be a cyclic group generated by a and $|a| = p$ then $\omega(\Gamma^o(C_p)) = 2$, ω stands for the clique number.*

Proof. Clique of a graph $\Gamma^o(C_p)$ contains all those vertices of the graph that produce a complete regular subgraph of $\Gamma^o(C_p)$. The cardinality of all such vertices is termed as a clique number of the graph. Further $\Gamma^o(C_p)$ is an order structured graph of order p. As each vertex is either of degree 1 or $p-1$. Ultimately, it produces $k_{1,p-1}$ a complete bipartite graph. And all $p-1$ non-trivial vertices are not adjacent to each other. And e is adjacent to $p-1$ vertices. Therefore the only complete subgraph of $\Gamma^o(C_p)$ that exists is a path graph of order two. Therefore,

$$\omega(\Gamma^o(C_p)) = |P_2| = 2$$

□

The spectral relation on $\Gamma^o(C_p)$ will help to classify the graphs by specifying certain graphical parameters as illustrated in (a) and (b) parts of Theorem 11 which relates to some special type of eigen values like Fiedler eigen values and the eigen values of the Laplacian matrix. And the (c) part states the degree of the neutral element.

Theorem 11. *Consider a periodic group G, $G \cong C_p, p \geq 3$ then in $\Gamma^o(C_p)$,*

(a) *Fiedler eigen value is always constant that is one.*

(b) *The eigen values of Laplacian matrices has a generalization,*

$$\begin{aligned}\Sigma|\lambda_i| &= 0+(p-2)+p\\ \Sigma|\lambda_i| &= 2(p-1).\end{aligned}$$

(c) $deg(e) = p-1$.

The energy of graph $\Gamma^o(C_p)$ depends upon the absolute sum of its eigen values. The following Proposition 1 make a connection between energy and the order of $\Gamma^o(C_p)$.

Proposition 1. *In $\Gamma^o(G)$ for $G \cong C_p$, $E(\Gamma^o(G)) = 2\sqrt{p-1}$ where E stands for the Energy of the graph.*

For the spanning tree, we now discuss all the connected subgraphs with specific order and size.

Theorem 12. *The number of spanning trees in an order structured graph $\Gamma^o(C_n)$ for a cyclic group of order n is identical to the number of unique order single vertices existing in the group.*

Proof. Let $\Gamma^o(C_n)$ be an order structured graph defined for a finite group generated by a single element. If a single element of unique order exists then by the definition of order structured graph on a group of symmetries, it is always adjacent to all the remaining vertices of $\Gamma^o(C_n)$.

If we consider a subgraph having a unique order vertex aside by all the linked edges then it generates a spanning tree. As the order of the subgraph is same as that of the cardinality of the vertex set of $\Gamma^o(C_n)$. In the same manner, the number of spanning trees is same as that of the number of unique order vertices. □

For an enchanting relation regarding the degree of any arbitrary vertex , the forthcoming result is crucial. We have linked the degree to p and to the Euler's Phi function.

Theorem 13. *If $a^k \in C_p$ in $\Gamma^o(C_p)$ then for a divisor d_i of $|G|$, we have*

$$deg(a^k) = |G| - \phi(d_i).$$

Proof. Let C_p be a cyclic group of a prime order p. And $\Gamma^o(C_p)$ represents the order structured graph on p vertices. Clearly, in cyclic groups corresponding to each d_i belongs to the divisors of p, there are exactly $\phi(d_i)$ elements having order d_i. Here p is prime so only two positive divisors 1 and p exists.

Case-1: If $d_i = 1$ then evidently for such a divisor there always exists at-least one element of that order. In this case e is the only element of order 1. Further we know that $deg(e) = p-1$ or $p-\phi(1) = p-\phi(d_i)$. Thus the result holds.

Case-2: Now if $d_i = p$ then essentially there will be an element of order p in C_p. As p is prime and C_p is a cyclic group so each non identity element of the group has co-prime exponent to p. Therefore $o(a^i) = p$ for $1 \leq i \leq p-1$. So $p-1$ non zero powered elements are of order p. Furthermore, the order structured simple graph affirms that these elements are not connected among themselves. So only adjacent vertex to every such element is e. Therefor $deg(a^k) = 1$ or alternatively,

$$deg(a^k) = p-\phi(p) = p-\phi(d_i).$$

□

Proposition 2 demonstrates the list of vertices with distances identical to radius or minimum eccentricity of $\Gamma^o(C_p)$.

Proposition 2. *The following results holds for the OSSGs.*

(a) *Center of* $\Gamma^o(C_p)$ *is always* $\{e\}$.
(b) *The Periphery for* $\Gamma^o(C_p)$ *is* $\{a, a^2, \ldots, a^{p-1}\}$

The Theorem 14 of this section is related with the domination number, a graphical parameter.

Theorem 14. *The domination number* γ *of* $\Gamma^o(C_p)$ *is always one.*

Proof. In $\Gamma^o(C_p)$, C_p is a group with only one generator. Now to explore the domination number of $\Gamma^o(C_p)$, we observe the cardinality of a minimum dominating set. Where a dominating set is a subset V_1 of the vertex set V of $\Gamma^o(C_p)$ such that each vertex in the complement set of V_1 is adjacent to some vertex in V_1. Let $V_1 = \{e\}$ then $V_1' = C_p - \{e\}$. That is V_1' contains all non identity elements. Also by the definition of order structured graph $\Gamma^o(C_p)$ produces a star graph. Therefore each non identity vertex in V_1' is definitely connected with V_1. Thus V_1 produces a smallest dominating set, consequently,

$$\gamma(\Gamma^o(C_p)) = |V_1| = 1$$

□

This section characterized the graph regarding the degree of $a^k \in C_p$ for $1 \leq k \leq p-1$, that is always lesser than the degree of e. The open neighbourhood of a^k is interlinked with e but the closed neighbourhood shares a^k too in the open neighbourhood. The subgraph of $\Gamma^o(C_p)$ isomorphic to $\Gamma^o(C_p)$ is the only spanning tree that does exist. The open neighbourhood of e belonging to the vertex set of $\Gamma^o(C_p)$ includes all the non-identity elements of C_p.

These graphs generated by a single element has clique number consistent to 2. Further, the graphs are classified in terms of various spectral parameters. The criterion for the calculation of the degree of any arbitrary vertex a^l, $1 \leq l \leq p-1$ is $p - \phi(d_i)$ with d_i dividing p, is explored. As e is the only element that is directly linked to each vertex of $\Gamma^o(C_p)$, so e lies in center of the graph.

4 Classification of $\Gamma^o(C_p)$ Through Some Special Graphs:

The graph $\Gamma^o(C_p)$ is classified under certain types of special graphs in this section. A relationship is developed between the order of the graph and path graphs. It is assured that the closure graph and the original constructed graph both coincides.

These order structured graphs are further classified into hamiltonian, eulerian, semi-eulerian and path graphs. The closure graph is the extension of the original graph. That is made by the construction of some extra edges based on sum of the degrees of each pair of vertices.

Theorem 15. *For* $\Gamma^o(C_p)$ *with* C_p *a prime order cyclic group, the closure graph remains isomorphic to* $\Gamma^o(C_p)$.

Proof. Let C_p be a group generated by a single element a. Also $\Gamma^o(C_p)$ be an order structured graph of C_p. Next to find out the closure graph of $\Gamma^o(C_p)$, we will observe those pair of non adjacent vertices l, m, the sum of whose degrees produces a number greater than or equal to p that is $deg(l) + deg(m) \geq p$. As C_p contains p elements so $\Gamma^o(C_p)$ has p vertices further for arbitrary vertex v_i of $\Gamma^o(C_p)$,

$$\deg(v_i) = \begin{cases} p-1 & \text{if } v_i = e, \\ 1 & \text{if } v_i \neq e. \end{cases}$$

So, $e \sim v_i$ for every non identity vertex v_i. Therefore, the pair of non-identity vertices v_i, v_j for $i \neq j$ are the only non adjacent vertices that remains to check for the closure graph of $\Gamma^o(C_p)$. But $deg(v_i) = 1 = deg(v_j)$. It further implies that

$$deg(v_i) + deg(v_j) = 2 < p.$$

So, no more edges are added to $\Gamma^o(C_p)$. Ultimately, we get the closure graph of $\Gamma^o(C_p)$ isomorphic to $\Gamma^o(C_p)$. □

Theorem 16. *The removal of a single vertex from $\Gamma^o(C_p)$ makes the graph disconnected.*

Proof. Let $\Gamma^o(C_p)$ be an order structured graph of C_p. Also $\Gamma^o(C_p)$ being a star graph has $deg(e) = p - 1$ and $deg(a^i) = 1, \forall\, 1 \leq i \leq p - 1$. Further in our case to observe the presence of a cut vertex we use the strategy of removing the vertex with the highest degree.

As the highest degree vertex being of having maximum number of linked edges, has the greatest likelihood to disconnect the graph. Here e is the maximum degree vertex, if we remove it then all the linked edges will be removed. Consequently, we deduce that e is the cut vertex of $\Gamma^o(C_p)$. As the removal of e breaks the graph in components. Ultimately, $\Gamma^o(C_p)$ has e as the cut vertex. □

Theorem 17. *$\Gamma^o(C_p)$ is always a separable graph.*

Proof. Let $\Gamma^o(C_p)$ is constructed for a cyclic group C_p. It is deduced that $\Gamma^o(C_p)$ produces a star graph. And the vertex connectivity of a star graph is always one. That is the minimum number of vertices required to remove to disconnect $\Gamma^o(C_p)$ is one only that can never be a leaf vertex. Therefore $\Gamma^o(C_p)$ is a separable graph as it contains a single cut vertex whose removal disconnects the order structured graph. □

A remark is stated below, specifying the constrained graph in terms of path graphs.

Remark 1. *$\Gamma^o(C_2)$ and $\Gamma^o(C_3)$ are always path graphs.*

Hamiltonian nature of $\Gamma^o(C_p)$ is explained in the succeeding theorem.

Theorem 18. *$\Gamma^o(C_p)$ an order structured graph of C_p can never be a hamiltonian graph.*

Proof. Let $C_p =< a, a^p = e >$ be a cyclic group and $\Gamma^o(C_p)$ represents an order structured simple graph of C_p. Now for $\Gamma^o(C_p)$ to be hamiltonian there must exists a hamiltonian cycle in $\Gamma^o(C_p)$. And the hamiltonian cycle is a cycle which visits every single vertex exactly once and does not contain any repeated vertex. But in $\Gamma^o(C_p)$ only $e \sim a^i$ for each non-identity element a^i of $\Gamma^o(C_p)$. Further $a^i \nsim a^j$ for every $1 \leq i, j \leq p-1$. That is there is no closed path among e, a^i and a^j, thus these vertices can not be traversed by any sequence of vertices.

Consequently we are unable to get a hamiltonian cycle. It further implies that $\Gamma^o(C_p)$ can never be a hamiltonian graph for any choice of p. □

Corollary 2. *$\Gamma^o(C_p)$ is neither eulerian nor semi-eulerian for any choice of p.*

Theorem 19. *The rank of OSSG $\Gamma^o(C_p)$ is 2 and its nullity is $p-2$.*

Proof. The rank of the adjacency matrix of $\Gamma^o(C_p)$, say P, can be discussed by stating the maximum number of non-zero rows in the reduced echelon form of P. The adjacency matrix has $a_{1j}=1$ for $2 \leq j \leq p$ and $a_{i1}=1$ for $2 \leq i \leq p$ with all the remaining entries of order p matrix is 0.

Now for the reduction of matrix P into reduced echelon form, firstly we subtract R_p and R_{p-1} that produces a zero row R_p. In the similar manner we continue subtracting the consecutive preceding rows ultimately R_1 and R_2 remains as $a_{11}=0$ and $a_{21}=1$. The subtraction can't produce a zero entry in any way. So the rank of the graph $\Gamma^o(C_p)$ is 2. The complement of the rank that is $p-2$ thus refers to the nullity of the graph $\Gamma^o(C_p)$. □

It is explored that $\Gamma^o(C_p)$ becomes a disconnected graph if we remove only a single vertex e from the graph. The order structured simple graph don't satisfy the properties of hamiltonian graph, eulerian and semi-eulerian graphs. The constructed graph can't be extended under the definition of closure graphs. Last but not the least, $\Gamma^o(C_p)$ is classified under the concepts of rank and nullity.

5 Discussion and conclusions

In coding theory, error correcting codes are framed with the help of algebraic structures. Many algebraic techniques are used in control system too.

It is evident from the algebraic structure $\Gamma^o(C_p)$ that the closed neighborhood of e is identical to C_p. Also the edge covering number is constant that is always one. The general criterion for the degree of any non-identity vertex of C_p is also explored.

Furthermore center, independence number, clique number and much more graphical parameters are discussed. And it is admitted that $\Gamma^o(C_p)$ can never be eulerian, semi-eulerain and hamiltonian.

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