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Improved Mean Estimators for Population Utilizing Dual Supplementary Characteristics Under Simple Random Sampling

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Abstract This paper makes another addition to the existing literature of population mean estimation. An improved family of mean estimators for the population is suggested using simple random sampling utilizing the information of dual supplementary characteristics. We have conducted an extensive theoretical and numerical investigation of these recommended estimators using the criteria of bias and mean square error. The study provides derivations of the bias and mean square error of the recommended estimators, approximating up to the first order and compared with the existing estimators. The suggested estimators are also studied numerically and compared using real-world data sets and simulation studies. The theoretical and numerical comparisons show that the estimators suggested in the study are much more efficient than the competitor estimators under all situations i.e. Bahl and Tuteja [5] and other existing estimators in the literature. Therefore, the suggested family of estimators can be applied to real-life problems to obtain better results than the existing mean estimators for the population.

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1 Introduction

It happens frequently that the supplementary information contributes significantly in improving the precision for the population estimators. This increase in precision is due to benefiting the advantage of connection between the research variable and the supplementary variable. Different estimators like regression, ratio, exponential and product etc. have been proposed by many authors using supplementary information under different sampling designs. Cochran [7] derived a new ratio estimator for the mean using auxiliary information. Bhushan et al. [6] and Ahmad et al.

[2] also proposed many efficient estimators with the help of different sampling techniques. Subramani and Kumarapandiyan [19] started work by developing an enhanced ratio estimator for the estimation of a population mean utilizing different parameters of the concomitant variable. Subzar et al. [20] used population deciles and correlation coefficient of the auxiliary variables. Subzar et al. [21] also suggested a class of new estimators with the help of supplementary information of population median, deciles and their linear combination with coefficient of variation and correlation coefficient.

There arise situations when the supplementary information exists in attributes form, which is correlated highly with the variable under study. Taking into account, relationship between the study variable and supplementary characteristic, many authors including Naik and Gupta [13], Lone et al. [11], Kumar and Singh [9] and Rather et al. [14] paid attention by using ratio estimators to improve the estimation of the population parameter. Singh et al. [16] developed several populations mean estimators of the variable under study with the help of the population proportion of the supplementary attribute. Kumar and Saini [8] introduced a predictive approach for the finite population mean estimation under auxiliary attribute. Zaman and Kadilar [23] developed a novel family of estimators using the information of supplementary characteristics. Ahmad et al. [3] and Yunusa [4] developed a class of general type estimators for the parameters of a population using the information of auxiliary attributes. Saini et al. [15] suggested optimum estimators under dual supplementary attributes with application in education, fisheries and agriculture areas. Wynn [22] proposed ratio mean estimator for population employing the characters of auxiliary variable in population proportion form. For further research, Singh et al. [18] developed a ratio of proportion using the auxiliary variables.

In this paper, a family of enhance mean estimators for population utilizing the known parameters of the supplementary attributes, for example coefficient of variation, quartile deviation, quartiles and median. The bias as well as the MSE of the recommended enhanced estimators are derived up to the approximation of first order. Based on theoretical and numerical comparisons, the applications of the proposed family estimator are elaborated. Therefore, the suggested novel estimators have the potential to be helpful in a variety of applications, and it offers a fresh and significant contribution regarding the estimation of population mean.

2 *Notations*

Consider a finite population $\psi = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_N\}$ of N identifiable elements. Using simple random sampling, n items of a sample are selected without replacement form the population ψ. Let y_i denotes the values of the study variable Y and $\pi_{ki}, (k = 1, 2)$ shows the characteristic of the supplementary attribute π_k, based on i^{th} element of population. It is assumed that $\pi_{ki} = 1$ if i^{th} unit of the population possess the attribute, and $\pi_{ki} = 0$, if otherwise. Let $G_k = \sum_{i=1}^{N} \pi_{ki}$ and $g_k = \sum_{i=1}^{n} \pi_{ki}$ denotes the overall population and sample units respectively, while $T_k = \frac{G_k}{N}$ and $t_k = \frac{g_k}{n}$ be the population and sample proportion. Let $\bar{y}$ and $\bar{Y}$ respectively denote the sample and population mean of the variable under study whereas sample variance $s_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2$ be the unbiased estimator of the population variance $S_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^2$. The variance of the auxiliary attributes of sample and population respectively denoted by $s_{\pi_k}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\pi_{ki} - T_k\right)^2$ and $S_{\pi_k}^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(\pi_{ki} - T_k\right)^2$. Let $C_Y = \frac{S_Y}{\bar{Y}}$ and $C_{\pi_k} = \frac{S_{\pi_k}}{T_k}$ shows the coefficient of variations of the study and the auxiliary attributes respectively. $\rho_{Y\pi_1}, \rho_{Y\pi_2}$ and $\rho_{\pi_1\pi_2}$ are the correlation coefficients between $(Y, \pi_1), (Y, \pi_2)$, and (π_1, π_2). To derive bias and MSE of the suggested estimators, let e_0, e_1 and e_2 be the error terms related to variable Y and attributes π_1 and π_2 respectively.

$$e_0 = \left(\frac{\bar{y} - \bar{Y}}{\bar{Y}}\right), e_1 = \left(\frac{t_1 - T_1}{T_1}\right) \text{ and } e_2 = \left(\frac{t_2 - T_2}{T_2}\right) \text{ whereas } E\left(e_0\right) = E\left(e_1\right) = E\left(e_2\right) = 0$$

and some expectations under SRS are given as:

$$E\left(e_0^2\right) = \lambda_0 C_y^2 = F_{200}, \quad E\left(e_1^2\right) = \lambda_0 C_{\pi_1}^2 = F_{020}, \quad E\left(e_2^2\right) = \lambda_0 C_{\pi_2}^2 = F_{002}$$
$$E\left(e_0 e_1\right) = \lambda_0 \rho_{y\pi_1} C_y C_{\pi_1} = F_{110}, \quad E\left(e_0 e_2\right) = \lambda_0 \rho_{y\pi_2} C_y C_{\pi_2} = F_{101}, \text{ and } E\left(e_1 e_2\right) = \lambda_0 \rho_{\pi_1\pi_2} C_{\pi_1} C_{\pi_2} = F_{011},$$

where $f = \frac{n}{N}$ is the sampling fraction and $\lambda_0 = \frac{1-f}{n}$: is the correction factor of the population.

$$\rho_{y\pi_1} = \frac{C_{y\pi_1}}{C_y C_{\pi_1}}, \quad \rho_{y\pi_2} = \frac{C_{y\pi_2}}{C_y C_{\pi_2}}, \quad \rho_{\pi_1\pi_2} = \frac{C_{\pi_1\pi_2}}{C_{\pi_1} C_{\pi_2}}$$

2. Literature Review

In this section, we discuss some latest existing estimators of population mean using simple random sampling.

(1) The traditional population mean estimator $\bar{u}_0$, given as:

$$\bar{u}_0 = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

The variance of $\bar{u}_0$ is given by:

$$\operatorname{var}\left(\bar{u}_0\right) = \lambda_0 \bar{Y}^2 C_y^2 \tag{2}$$

(2) Naik and Gupta [13] developed the following ratio and product type estimators $\bar{u}_1$ and $\bar{u}_2$ given as

$$\bar{u}_1 = \bar{y}\left(\frac{T_1}{t_1}\right) \tag{3}$$
$$\bar{u}_2 = \bar{y}\left(\frac{t_1}{T_1}\right) \tag{4}$$

The bias and mean square error of $\bar{u}_1$ and $\bar{u}_2$, are provided as:

$$\text{Bias}\left(\bar{u}_1\right) = \lambda_0 \bar{Y}\left(C_{\pi_1}^2 - \rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{5}$$
$$\text{Bias}\left(\bar{u}_2\right) = \lambda_0 \bar{Y} \rho_{Y\pi_1} C_Y C_{\pi_1} \tag{6}$$

and

$$\mathrm{MSE}(\bar{u}_1) = \lambda_0 \bar{Y}^2 \left(C_y^2 + C_{\pi_1} - 2\rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{7}$$

$$\mathrm{MSE}(\bar{u}_2) = \lambda_0 \bar{Y}^2 \left(C_y^2 + C_{\pi_1} + 2\rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{8}$$

(3) Bahl and Tuteja [5] proposed the following exponential ratio and product of exponential type estimators as:

$$\bar{u}_3 = \bar{y} \exp\left(\frac{T_1 - t_1}{T_1 + t_1}\right) \tag{9}$$

$$\bar{u}_4 = \bar{y} \exp\left(\frac{t_1 - T_1}{t_1 + T_1}\right) \tag{10}$$

The above estimators' bias and mean square error are given by:

$$\mathrm{Bias}\left(\bar{u}_3\right) = \lambda_0 \bar{Y} \left(\frac{3}{8} C_{\pi_1}^2 - \frac{1}{2} \rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{11}$$

$$\mathrm{Bias}(\bar{u}_4) = \frac{1}{2} \lambda_0 \bar{Y} \left(\rho_{Y\pi_1} C_Y C_{\pi_1} - \frac{1}{4} C_{\pi_1}^2\right) \tag{12}$$

and

$$\mathrm{MSE}\left(\bar{u}_3\right) \cong \lambda_0 \bar{Y}^2 \left(C_y^2 + \frac{1}{4} C_{\pi_1}^2 - \rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{13}$$

$$\mathrm{MSE}(\bar{u}_4) \cong \lambda_0 \bar{Y}^2 \left(C_y^2 + \frac{1}{4} C_{\pi_1}^2 + \rho_{Y\pi_1} C_Y C_{\pi_1}\right) \tag{14}$$

(4) The Kumar and Bhougal [10] developed an exponential estimator for population mean, given as

$$\bar{u}_5 = \bar{y}\left[\gamma_0 \exp\left(\frac{T_1 - t_1}{T_1 + t_1}\right) + (1 - \gamma_0) \exp\left(\frac{t_1 - T_1}{t_1 + T_1}\right)\right] \tag{15}$$

where γ is unknown constant whose value is $\gamma_{0(opt)} \cong \frac{1}{2} + \rho_{Y\pi_1} \frac{C_Y}{C_{\pi_2}}$

The estimator's $\bar{u}_5$ bias and MSE is given by:

$$\mathrm{Bias}\left(\bar{u}_5\right) = \lambda_0 \bar{Y} \left(\frac{1}{8}\left(4\gamma_0 - 1\right) C_{\pi_1}^2 - \left(\gamma_0 - \frac{1}{2}\right) \rho_{Y\pi_1} C_Y C_{\pi_2}\right) \textit{ and} \tag{16}$$

$$\mathrm{MSE}\left(u_5\right) \cong \lambda_0 \bar{Y}^2 C_y^2 \left(1 - \rho_{Y\pi_1}^2\right) \tag{17}$$

and

(5) The Sing and Kumar [17] developed ratio and product dual type estimators are given as:

$$\bar{u}_6 = \bar{y}\left(\frac{T_1}{t_1}\right)\left(\frac{T_2}{t_2}\right) \tag{18}$$

$$\bar{u}_7 = \bar{y}\left(\frac{t_1}{T_1}\right)\left(\frac{t_2}{T_2}\right) \tag{19}$$

The above estimator's bias and mean square error are given by:

$$\text{Bias}\left(\bar{u}_6\right) = \lambda_0\bar{Y}\left(C_{\pi_1}^2 + C_{\pi 2}^2 + \rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2} - \rho_{y\pi_1}C_yC_{\pi_1} - \rho_{y\pi_2}C_yC_{\pi_2}\right) \tag{20}$$

$$\text{Bias}\left(\bar{u}_7\right) = \lambda_0\bar{Y}\left(\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2} + \rho_{y\pi_1}C_yC_{\pi_1} + \rho_{y\pi_2}C_yC_{\pi_2}\right) \tag{21}$$

and

$$\text{MSE}\left(\bar{u}_6\right) \cong \lambda_0\bar{Y}^2\left[C_Y^2 + C_{\pi_1}^2 + C_{\pi 2}^2 - 2\left(\rho_{y\pi_1}C_yC_{\pi_1} - \rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2} + \rho_{y\pi_2}C_yC_{\pi_2}\right)\right] \tag{22}$$

$$\text{MSE}\left(\bar{u}_7\right) \cong \lambda_0\bar{Y}^2\left[C_Y^2 + C_{\pi_1}^2 + C_{\pi 2}^2 + 2\left(\rho_{y\pi_1}C_yC_{\pi_1} + \rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2} + \rho_{y\pi_2}C_yC_{\pi_2}\right)\right] \tag{23}$$

(6) The Ahmad et al. [1] proposed different exponential type mean estimators using auxiliary attributes given as:

$$\bar{u}_8 = \bar{y}\exp\left(\frac{T_1 - t_1}{T_1 + t_1}\right)\exp\left(\frac{T_2 - t_2}{T_2 + t_2}\right) \tag{24}$$

$$\bar{u}_9 = \bar{y}\exp\left(\frac{T_1 - t_1}{T_1 + t_1}\right)\exp\left(\frac{t_2 - T_2}{t_2 + T_2}\right) \tag{25}$$

$$\bar{u}_{10} = \bar{y}\exp\left(\frac{T_1 - t_1}{T_1 + t_1}\right)\exp\left[\frac{n\left(T_2 - t_2\right)}{2NT_2 - n\left(T_2 + t_2\right)}\right] \tag{26}$$

The biases of the estimators $\bar{u}_8, \bar{u}_9$ and $\bar{u}_{10}$ are given by:

$$\text{Bias}\left(\bar{u}_8\right) = \lambda_0\bar{Y}\left(\frac{3}{8}C_{\pi_1}^2 + \frac{3}{8}C_{\pi_2}^2 - \frac{1}{2}\rho_{Y\pi_1}C_yC_{\pi_1} - \frac{1}{2}\gamma_2\rho_{y\pi_2}C_yC_{\pi_2}\right) \tag{27}$$

$$\text{Bias}\left(\bar{u}_9\right) = \lambda_0\bar{Y}\left(-\frac{1}{8}C_{\pi_1}^2 - \frac{1}{8}C_{\pi_2}^2 - \frac{1}{2}\rho_{Y\pi_1}C_yC_{\pi_1}\right) \tag{28}$$

$$\text{Bias}\left(\bar{u}_{10}\right) = \lambda_0\bar{Y}\left(\frac{3}{8}C_{\pi_1}^2 + \frac{1}{8}C_{\pi_2}^2 - \frac{1}{4}\left(\frac{f}{1-f}\right)^2 C_{\pi_2}^2 - \frac{1}{2}\rho_{Y\pi_1}C_yC_{\pi_1} - \frac{1}{2}\left(\frac{f}{1-f}\right)\rho_{Y\pi_2}C_yC_{\pi_2}\right) \tag{29}$$

The MSEs of the estimator $\bar{u}_8, \bar{u}_9$ and $\bar{u}_{10}$ are given by:

$$\text{MSE}\left(\bar{u}_8\right) \cong \lambda_0\bar{Y}^2\left(C_Y^2 + \frac{1}{4}C_{\pi_1}^2 + \frac{1}{4}C_{\pi 2}^2 - \rho_{y\pi_1}C_yC_{\pi_1} - \rho_{y\pi_2}C_yC_{\pi_2} + \frac{1}{2}\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2}\right) \tag{30}$$

$$\text{MSE}\left(\bar{u}_9\right) \cong \lambda_0\bar{Y}^2\left(C_Y^2 + \frac{1}{4}C_{\pi_1}^2 + \frac{1}{4}C_{\pi 2}^2 - \rho_{y\pi_1}C_yC_{\pi_1} + \rho_{y\pi_2}C_yC_{\pi_2} - \frac{1}{2}\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2}\right) \tag{31}$$

$$\text{MSE}\left(\bar{u}_{10}\right) \cong \lambda_0\bar{Y}^2\left[C_Y^2 + \frac{1}{4}C_{\pi_1}^2 + \frac{1}{4}\left(\frac{f}{1-f}\right)^2 C_{\pi 2}^2 - \rho_{y\pi_1}C_yC_{\pi_1} - \left(\frac{f}{1-f}\right)\rho_{y\pi_2}C_yC_{\pi_2} + \frac{1}{2}\left(\frac{f}{1-f}\right)\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2}\right] \tag{32}$$

3. Proposed Family of Estimators

Motivated by the studies of Kumar and Bhougal [10], Singh and Kumar [17] and Ahmad et al. [1] optimum family of exponential mean estimators $\bar{u}_{Pj}$ for population under supplementary attributes is suggested as:

$$\bar{u}_{\mathrm{pj}} = \bar{y}\left(\frac{\omega_1 T_1 + \omega_2}{\omega_1 t_1 + \omega_2}\right)^{r_1}\left(\frac{\omega_1 T_2 + \omega_2}{\omega_1 t_2 + \omega_2}\right)^{r_2} \exp\left(\frac{\omega_3 (T_1 - t_1)}{\omega_3 (T_1 + t_1) + 2\omega_4}\right) \exp\left(\frac{\omega_3 (T_2 - t_2)}{\omega_3 (T_2 + t_2) + 2\omega_4}\right) \quad (33)$$

where r_1 and r_2 are constants which are unknown and its values are to be obtained to optimized the mean square error $\bar{u}_{\mathrm{pj}}$. The terms $\omega_1, \omega_2, \omega_3$ and ω_4 are the functions and identified real values of the auxiliary attributes. The suggested optimum estimators $\bar{u}_{\mathrm{pj}}$ in terms of e_0, e_1 and e_2 are described as follows:

$$\bar{u}_{\mathrm{pj}} = \bar{Y}(1 + e_0)\left(\frac{\omega_1 T_1 + \omega_2}{\omega_1 T_1 (1 + e_1) + \omega_2}\right)^{r_1}\left(\frac{\omega_1 T_2 + \omega_2}{\omega_1 T_2 (1 + e_2) + \omega_2}\right)^{r_2} \exp\left(\frac{\omega_3 (T_1 - T_1 (1 + e_1))}{\omega_3 (T_1 + T_1 (1 + e_1)) + 2\omega_4}\right) \exp\left(\frac{\omega_3 (T_2 - T_2 (1 + e_2))}{\omega_3 (T_2 + T_2 (1 + e_2)) + 2\omega_4}\right) \quad (34)$$

Simplifying the above equation, we have

$$\bar{u}_{\mathrm{pj}} = \bar{Y}(1 + e_0)\left(\frac{1}{1 + \theta_1 e_1}\right)^{r_1}\left(\frac{1}{1 + \theta_2 e_2}\right)^{r_2} \exp\left[\frac{-\theta_3 e_1}{2\left(1 + \frac{1}{2}\theta_3 e_1\right)}\right] \exp\left[\frac{-\theta_4 e_2}{2\left(1 + \frac{1}{2}\theta_4 e_2\right)}\right] \quad (35)$$

where $\theta_1 = \frac{\omega_1 T_1}{\omega_1 T_1 + \omega_2}, \theta_2 = \frac{\omega_1 T_2}{\omega_1 T_2 + \omega_2}, \theta_3 = \frac{\omega_3 T_1}{\omega_3 T_1 + \omega_4}$ and $\theta_4 = \frac{\omega_3 T_2}{\omega_3 T_2 + \omega_4}$

Using Taylor Expansion and ignoring terms of higher order, we get:

$$\bar{u}_{p_j} = \bar{Y}\left[\begin{array}{r} 1 - \left(\theta_2 r_2 + \frac{\theta_4}{2}\right) e_2 + \left(\theta_2^2 r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2 r_2}{2}\right) e_2^2 - \left(\theta_1 r_1 + \frac{\theta_3}{2}\right) e_1 + \left(\theta_1 \theta_2 r_1 r_2 + \frac{\theta_3 \theta_4 r_1}{2} + \frac{\theta_3 \theta_4 r_2}{2}\right) e_1 e_2 + \\ \left(\theta_1^2 r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2 r_1}{2}\right) e_1^2 + e_0 - \left(\theta_2 r_2 + \frac{\theta_4}{2}\right) e_0 e_2 - \left(\theta_1 r_1 + \frac{\theta_3}{2}\right) e_0 e_1 \end{array}\right], \quad (36)$$

Subtracting $\bar{Y}$ from equation (36) both sides, we get,

$$\bar{u}_{Pj} - \bar{Y} = \bar{Y}\left[\begin{array}{r} -\left(\theta_2 r_2 + \frac{\theta_4}{2}\right) e_2 + \left(\theta_2^2 r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2 r_2}{2}\right) e_2^2 - \left(\theta_1 r_1 + \frac{\theta_3}{2}\right) e_1 + \left(\theta_1 \theta_2 r_1 r_2 + \frac{\theta_3 \theta_4 r_1}{2} + \frac{\theta_3 \theta_4 r_2}{2}\right) e_1 e_2 + \\ \left(\theta_1^2 r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2 r_1}{2}\right) e_1^2 + e_0 - \left(\theta_2 r_2 + \frac{\theta_4}{2}\right) e_0 e_2 - \left(\theta_1 r_1 + \frac{\theta_3}{2}\right) e_0 e_1 \end{array}\right], \quad (37)$$

Applying expectation, the bias $\bar{u}_{Pj}$ is defined as:

$$\mathrm{Bias}\left(\bar{u}_{Pj}\right) = \bar{Y}\left[\begin{array}{r} \left(\theta_2^2 r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2 r_2}{2}\right) F_{002} + \left(\theta_1 \theta_2 r_1 r_2 + \frac{\theta_3 \theta_4 r_1}{2} + \frac{\theta_3 \theta_4 r_2}{2}\right) F_{011} + \left(\theta_1^2 r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2 r_1}{2}\right) F_{020} \\ -\left(\theta_2 r_2 + \frac{\theta_4}{2}\right) F_{101} - \left(\theta_1 r_1 + \frac{\theta_3}{2}\right) F_{110} \end{array}\right], \quad (38)$$

To obtain mean square error of $\bar{u}_{Pj}$, squaring both sides of equation (37), we get:

$$\left(\bar{u}_{P_j}-\bar{Y}\right)^2=\left[\bar{Y}\left\{\begin{array}{r}-\left(\theta_2 r_2+\frac{\theta_4}{2}\right)e_2+\left(\theta_2^2 r_2^2+\frac{3\theta_4^2}{8}+\frac{\theta_4^2 r_2}{2}\right)e_2^2-\left(\theta_1 r_1+\frac{\theta_3}{2}\right)e_1+\left(\theta_1\theta_2 r_1 r_2+\frac{\theta_3\theta_4 r_1}{2}+\frac{\theta_3\theta_4 r_2}{2}\right)e_1 e_2+\\ \left(\theta_1^2 r_1^2+\frac{3r_3^2}{8}+\frac{\theta_3^2 r_1}{2}\right)\theta_1^2 e_1^2+e_0-\left(\theta_2 r_2+\frac{\theta_4}{2}\right)e_0 e_2-\left(\theta_1 r_1+\frac{\theta_3}{2}\right)e_0 e_1\end{array}\right]^2\right], \tag{39}$$

Solving equation (39), the MSE of $\bar{u}_{Pj}$ is given as:

$$\text{MSE}\left(\bar{u}_{P_j}\right)\cong\bar{Y}^2\left[\begin{array}{r}\left(\theta_2 r_2+\frac{\theta_4}{2}\right)^2 F_{002}+\left(\theta_1 r_1+\frac{\theta_3}{2}\right)^2 F_{020}+F_{200}+2\left(\theta_2 r_2+\frac{\theta_4}{2}\right)\left(\theta_1 r_1+\frac{\theta_3}{2}\right)F_{011}-\\ 2\left(\theta_2 r_2+\frac{\theta_4}{2}\right)F_{101}-2\left(\theta_1 r_1+\frac{\theta_3}{2}\right)F_{110}\end{array}\right], \tag{40}$$

$$\text{MSE}\left(\bar{u}_{P_j}\right)\cong\bar{Y}^2\left[A_2^2F_{002}+A_1^2F_{020}+F_{200}+2\left(A_1A_2F_{011}-A_2F_{101}-A_1F_{110}\right)\right], \tag{41}$$

where $A_1=\left(\theta_1 r_1+\frac{\theta_3}{2}\right)$ and $A_2=\left(\theta_2 r_2+\frac{\theta_4}{2}\right)$.

3.1 Optimality of r_1 and r_2

To obtain the optimum values of r_1 and r_2, minimizing the $MSE\left(\bar{u}_{P_j}\right)$ in equaation (39) with respect to r_1 and r_2, We have $\frac{\partial}{\partial r_1}MSE\left(\bar{u}_{Pj}\right)=0$

$$\frac{\partial}{\partial r_1}\bar{Y}^2\left[A_2^2F_{002}+A_1^2F_{020}+F_{200}+2\left(A_1A_2F_{011}-A_2F_{101}-A_1F_{110}\right)\right]=0$$

and $\frac{\partial}{\partial r_2}$ MSE $\left(\bar{u}_{Pj}\right)=0$

$$\frac{\partial}{\partial r_2}\bar{Y}^2\left[A_2^2F_{002}+A_1^2F_{020}+F_{200}+2\left(A_1A_2F_{011}-A_2F_{101}-A_1F_{110}\right)\right]=0$$

The optimum expressions of r_1 and r_2 are:

$$r_{1(opt)}=\frac{\frac{F_{110}}{\theta_1F_{020}}-\frac{F_{101}F_{011}}{\theta_1F_{020}F_{002}}+\frac{\theta_3F_{011}^2}{2\theta_1F_{020}F_{002}}-\frac{\theta_3}{2\theta_1}}{1-\frac{F_{011}^2}{F_{020}F_{002}}}\text{ and }r_{2(opt)}=\frac{\frac{F_{101}}{\theta_2F_{002}}-\frac{F_{110}F_{011}}{\theta_2F_{020}F_{002}}+\frac{\theta_4F_{011}^2}{2\theta_2F_{020}F_{002}}-\frac{\theta_4}{2\theta_2}}{1-\frac{F_{011}^2}{F_{020}F_{002}}}.$$

Now replacing r_1 and r_2 in the values of A_1 and A_2 respectively, by optimum values $r_{1(\text{ opt})}$ and $r_{2(\text{ opt})}$ The optimum MSE of the suggested estimators is given as:

$$\text{MSE}\left(\bar{u}_{P_j}\right)_{opt}\cong\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_4F_{101}-A_3F_110\right)\right], \tag{42}$$

where $A_3=\left(\theta_1 r_1(opt)+\frac{\theta_3}{2}\right)$ and $A_4=\left(\theta_2 r_2(opt)+\frac{\theta_4}{2}\right)$.

The family of estimators is defined using different choices of $\omega_1,\omega_2,\omega_3$ and ω_4 (i.e. $C_{\pi}1, C_{\pi}2$, T_1, $T_2, \rho_{Y_{\pi_1}}, \rho_{Y_{\pi_2}}$,.... $\rho_{\pi_1\pi_2}, C_{\pi_1}, C_{\pi_2}$ in equation (33). A total of four expressions for each $\theta_1,\theta_2,\theta_3$ and θ_4 are obtained for the proposed population mean estimators using auxiliary attributes in SRS. Using different choices of auxiliary attributes, different estimators, their biases and MSEs are provided in Section 3.2.

3.2 Different Choices of $\omega_1, \omega_2, \omega_3$ and ω_4 :

Choice I: For $\omega_1 = C_{\pi_2}; \omega_2 = T_2; \omega_3 = \rho^3_{Y_{\pi_1}}; \omega_4 = C^3_{\pi_2}$
Putting these values in equation (33), we have:

$$\bar{u}_{PI} = \bar{y}\left(\frac{C_{\pi_2}T_1 + T_2}{C_{\pi_2}t_1 + T_2}\right)^{r_{1(opt)}}\left(\frac{C_{\pi_2}T_2 + T_2}{C_{\pi_2}T_t + T_2}\right)^{r_{2\,(opt)}} \exp\left(\frac{\rho^3_{Y_3}(T_1 - t_1)}{\rho^3_{Y\pi_3}(T_1 + t_1) + 2C^3_{\pi_2}}\right)\exp\left(\frac{\rho^3_{Y_{\pi_3}}(T_2 - t_2)}{\rho^3_{Y_{\pi_3}}(T_2 + t_2) + 2C^3_{\pi_2}}\right) \quad (43)$$

The bias of the estimator $\bar{u}_{P1}$ is given as:

$$\text{Bias}(\bar{u}_{P1}) = \bar{Y}\left(A_5F_{002} + A_6F_{011} + A_7F_{020} - A_8F_{101} - A_9F_{110}\right) \quad (44)$$

The MSE of the estimator $\bar{u}_{P1}$ is given as:

$$\text{MSE}(\bar{u}_{P1}) \cong \bar{Y}^2\left[A_8^2F_{002} + A_9^2F_{020} + F_{200} + 2\left(A_9A_8F_{011} - A_8F_{101} - A_9F_{110}\right)\right] \quad (45)$$

$$A_5 = \left(\theta_2^2r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2r_2}{2}\right), \quad A_6 = \left(\theta_1\theta_2r_1r_2 + \frac{\theta_3\theta_4r_1}{2} + \frac{\theta_3\theta_4r_2}{2}\right), \quad A_7 = \left(\theta_1^2r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2r_1}{2}\right)$$

$$A_8 = \left(\theta_2r_{2\,(opt)} + \frac{\theta_4}{2}\right) \text{ and } A_9 = \left(\theta_1r_{1\,(opt)} + \frac{\theta_3}{2}\right)$$

$$\theta_1 = \frac{C_{\pi_2}T_1}{C_{\pi_2}T_1 + T_2}, \theta_2 = \frac{C_{\pi_2}T_2}{C_{\pi_2}T_2 + T_2}, \theta_3 = \frac{\rho^3_{Y\pi_3}T_1}{\rho^3_{Y\pi_3}T_1 + C^3_{\pi_2}} \text{ and } \theta_4 = \frac{\rho^3_{Y\pi_3}T_2}{\rho^3_{Y\pi_3}T_2 + C^3_{\pi_2}}$$

Choice II: For $\omega_1 = T_2; \omega_2 = C_{\pi_2}; \omega_3 = \rho^3_{Y_1}; \omega_4 = C^3_{\pi_1}$
Putting these values in equation (33), we have:

$$\bar{u}_{P2} = \bar{y}\left(\frac{T_2T_1 + C_{\pi_2}}{T_2t_1 + C_{\pi_2}}\right)^{r_1(opt)}\left(\frac{T_2^2 + C_{\pi_2}}{T_2t_2 + C_{\pi_2}}\right)^{r_2(opt)} exp\left(\frac{\rho^3_{Y\pi_1}(T_1 - t_1)}{\rho^3_{Y\pi_1}(T_1 + t_1) + 2C^3_{\pi_1}}\right) exp\left(\frac{\rho^3_{Y\pi_1}(T_2 - t_2)}{\rho^3_{Y\pi_1}(T_2 + t_2) + 2C^3_{\pi_1}}\right) \quad (46)$$

The bias of the estimator $\bar{u}_{P2}$ is given as:

$$\text{Bias}(\bar{u}_{P2}) = \bar{Y}\left(A_{10}F_{002} + A_{11}F_{011} + A_{12}F_{020} - A_{13}F_{101} - A_{14}F_{110}\right) \quad (47)$$

The MSE of the estimator $\bar{u}_{P2}$ is given as:

$$\text{MSE}(\bar{u}_{P2}) \cong \bar{Y}^2\left[A_{13}^2F_{002} + A_{14}^2F_{020} + F_{200} + 2\left(A_{14}A_{13}F_{011} - A_{13}F_{101} - A_{14}F_{110}\right)\right] \quad (48)$$

$$A_{10} = \left(\theta_2^2r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2r_2}{2}\right), A_{11} = \left(\theta_1\theta_2r_1r_2 + \frac{\theta_3\theta_4r_1}{2} + \frac{\theta_3\theta_4r_2}{2}\right), A_{12} = \left(\theta_1^2r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2r_1}{2}\right)$$

$$A_{13} = \left(\theta_2r_{2(\,opt)} + \frac{\theta_4}{2}\right) \text{ and } A_{14} = \left(\theta_1r_{1(\,opt} + \frac{\theta_3}{2}\right)$$

$$\theta_1 = \frac{T_2T_1}{T_2T_1 + C_{\pi_2}}, \theta_2 = \frac{T_2T_2}{T_2^2 + C_{\pi_2}}, \theta_3 = \frac{\rho^3_{Y_1}T_1}{\rho^3_{Y_{\pi_1}}T_1 + C^3_{\pi_1}} \text{ and } \theta_4 = \frac{\omega_3T_2}{\omega_3T_2 + C^3_{\pi_1}}$$

Choice III: For $\omega_1 = C_{\pi_2}; \omega_2 = \rho_{\pi 1 \pi 2}; \omega_3 = \rho^3_{Y_{\pi_2}}; \omega_4 = C^3_{\pi 2}$
Putting these values in equation (33), we have:

$$\bar{u}_{P3} = \bar{y}\left(\frac{C_{\pi_2}T_1 + \rho_{\pi|\pi 2}}{C_{\pi_2}t_1 + \rho_{\pi|\pi 2}}\right)^{r_1(opt)}\left(\frac{C_{\pi_2}T_2 + \rho_{\pi|\pi 2}}{C_{\pi_2}t_2 + \rho_{\pi|\pi 2}}\right)^{r_2(opt)} exp\left(\frac{\rho^3_{Y_{\pi_2}}(T_1 - t_1)}{\rho^3_{Y_{\pi_2}}(T_1 + t_1) + 2C^3_{\pi_2}}\right) exp\left(\frac{\rho^3_{Y_{\pi_2}}(T_2 - t_2)}{\rho^3_{Y_{\pi_2}}(T_2 + t_2) + 2C^3_{\pi_2}}\right) \quad (49)$$

The bias of the estimator $\bar{u}_{P3}$ is given as:

$$Bias\left(\bar{u}_{p3}\right) = \bar{Y}\left(A_{15}F_{002} + A_{16}F_{011} + A_{17}F_{020} - A_{18}F_{101} - A_{19}F_{110}\right) \quad (50)$$

The MSE of the estimator $\bar{n}_{p3}$ is given as:

$$\text{MSE}\left(\bar{u_{p3}}\right) \cong \bar{Y}^2\left[A^2_{18}F_{002} + A^2_{19}F_{020} + F_{200} + 2\left(A_{19}A_{18}F_{011} - A_{18}F_{101} - A_{19}F_{110}\right)\right] \quad (51)$$

$$A_{15} = \left(\theta_2^2 r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2 r_2}{2}\right), A_{16} = \left(\theta_1\theta_2 r_1 r_2 + \frac{\theta_3\theta_4 r_1}{2} + \frac{\theta_3\theta_4 r_2}{2}\right), A_{17} = \left(\theta_1^2 r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2 r_1}{2}\right)$$

$$\text{A}_{18} = \left(\theta_2 r_{2(\text{ opt})} + \frac{\theta_4}{2}\right) \text{ and } A_{19} = \left(\theta_1 r_{1(\text{ opt}} + \frac{\theta_3}{2}\right)$$

$$\theta_1 = \frac{C_{\pi_2}T_1}{C_{\pi_2}T_1 + \rho_{\pi_1}\pi_2}, \theta_2 = \frac{C_{\pi_2}T_2}{C_{\pi_2}T_2 + \rho_{\pi_1}\pi_2}, \theta_3 = \frac{\rho^3_{Y_{\pi_2}}T_1}{\rho^3_{Y_{\pi_2}}T_1 + C^3_{\pi_2}} \text{ and } \theta_4 = \frac{\rho^3_{Y_{\pi_2}}T_2}{\rho^3_{Y_{\pi_2}}T_2 + C^3_{\pi_2}}$$

Choice IV: For $\omega_1 = \rho_{x1+1}; \omega_2 = T_2; \omega_3 = \rho^3_{Yr_1}; \omega_4 = C^3_{\pi 1}$
Putting these values in equation (33), we have:

$$\bar{u}_{p4} = \bar{y}\left(\frac{\rho_{\pi|\pi 2}T_1 + T_2}{\rho_{\pi|\pi 2}t_1 + T_2}\right)^{r_1(opt)}\left(\frac{\rho_{\pi|\pi 2}T_2 + T_2}{\rho_{\pi|\pi 2}t_2 + T_2}\right)^{r_2(opt)} exp\left(\frac{\rho^3_{Y_{\pi_2}}(T_1 - t_1)}{\rho^3_{Y_{\pi_2}}(T_1 + t_1) + 2C^3_{\pi_1}}\right) exp\left(\frac{\rho^3_{Y_{\pi_2}}(T_2 - t_2)}{\rho^3_{Y_{\pi_2}}(T_2 + t_2) + 2C^3_{\pi_1}}\right) \quad (52)$$

The bias of the estimator $\bar{u}_{p4}$ is given as:

$$Bias\left(\bar{u}_{p4}\right) = \bar{Y}\left(A_{20}F_{002} + A_{21}F_{011} + A_{22}F_{020} - A_{23}F_{101} - A_{24}F_{110}\right) \quad (53)$$

The MSE of the estimator $\bar{u}_{p4}$ is given as:

$$\text{MSE}\left(\bar{u}_{p4}\right) \cong \bar{Y}^2\left[A^2_{23}F_{002} + A^2_{24}F_{020} + F_{200} + 2\left(A_{24}A_{23}F_{011} - A_{23}F_{101} - A_{24}F_{110}\right)\right] \quad (54)$$

$$A_{20} = \left(\theta_2^2 r_2^2 + \frac{3\theta_4^2}{8} + \frac{\theta_4^2 r_2}{2}\right), A_{21} = \left(\theta_1\theta_2 r_1 r_2 + \frac{\theta_3\theta_4 r_1}{2} + \frac{\theta_3\theta_4 r_2}{2}\right), A_{22} = \left(\theta_1^2 r_1^2 + \frac{3\theta_3^2}{8} + \frac{\theta_3^2 r_1}{2}\right)$$

$$\text{A}_{23} = \left(\theta_2 r_{2(\text{ opt})} + \frac{\theta_4}{2}\right) \text{ and } A_{24} = \left(\theta_1 r_{1(\text{ opt}} + \frac{\theta_3}{2}\right)$$

$$\theta_1 = \frac{\rho_{\pi_1\pi_2}T_1}{\rho_{\pi_1\pi_2}T_1 + T_2}, \theta_2 = \frac{\rho_{\pi_1\pi_2}T_2}{\rho_{\pi_1\pi_2}T_2 + T_2}, \theta_3 = \frac{\rho^3_{Y_{\pi_2}}T_1}{\rho^3_{Y_{\pi_2}}T_1 + C^3_{\pi_1}} \text{ and } \theta_4 = \frac{\rho^3_{Y_{\pi_2}}T_2}{\rho^3_{Y_{\pi_2}}T_2 + C^3_{\pi_1}}$$

4. Theoretical efficiency comparisons

In this section, theoretical comparisons are made between the mean square error of the suggested estimators and rest of the existing estimators. The conditions for which the proposed estimators are more efficient than the existing estimators using the expressions of mean square error are given by:

Condition (i): MSE $\left(\bar{u}_{pj}\right)_{opt}$ < var $(\bar{u}_0)$, if

$$\lambda_0 \bar{Y}^2 C_y^2 - \bar{Y}^2 \left[A_4^2 F_{000} + A_3^2 F_{020} + F_{200} + 2\left(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{100}\right)\right] > 0$$

$$\text{or} \quad \lambda_0 C_y^2 - \left[A_4^2 F_{000} + A_3^2 F_{020} + F_{200} + 2\left(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{100}\right)\right] > 0'$$

Condition (ii): $MSE\left(\bar{u}_{pj}\right)_{opt} < MSE(\bar{u}_1)$, if

$$\lambda_0 \bar{Y}^2 \left(C_y^2 + C_{\pi_1} - 2\rho_{Y\pi_1} C_Y C_{\pi_1}\right) - \bar{Y}^2 [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

$$\text{or } \lambda_0 \left(C_y^2 + C_{\pi_1} - 2\rho_{Y\pi_1} C_Y C_{\pi_1}\right) - [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

Condition (iii): MSE $\left(\bar{u}_{pj}\right)_{\text{opt}}$ < MSE $(\bar{u}_2)$, if

$$\lambda_0 \bar{Y}^2 \left(C_5^2 + C_n + 2\rho_{Y_n} C_7 C_{n9}\right) - \bar{Y}^2 \left[A_4^2 F_{002} + A_3^2 F_{020} + F_{30} + 2\left(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110}\right)\right] > 0$$

$$\text{or } \lambda_0 \left(C_y^2 + C_{x_1} + 2\rho_{V_1} C_\gamma C_{x_1}\right) - \left[A_4^2 F_{020} + A_5^2 F_{000} + F_{200} + 2\left(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{100}\right)\right] > 0$$

Condition(iv): MSE $\left(\bar{u}_{pj}\right)_{opt} < MSE\ (\bar{u}_3)$, if

$$\lambda_0 \bar{Y}^2 \left(C_y^2 + \frac{1}{4} C_{\pi_1} - \rho_{Y\pi_1} C_Y C_{\pi_1}\right) - \bar{Y}^2 [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

$$\text{or } \lambda_0 \left(C_y^2 + \frac{1}{4} C_{\pi_1} - \rho_{Y\pi_1} C_Y C_{\pi_1}\right) - [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

Condition (v): $MSE\left(\bar{u}_{pj}\right)_{opt} < MSE(\bar{u}_4)$, if

$$\lambda_0 \bar{Y}^2 \left(C_y^2 + \frac{1}{4} C_{\pi_1} + \rho_{Y\pi_1} C_Y C_{\pi_1}\right) - \bar{Y}^2 [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

$$\text{or } \lambda_0 \left(C_y^2 + \frac{1}{4} C_{\pi_1} + \rho_{Y\pi_1} C_Y C_{\pi_1}\right) - [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

Condition (vi): MSE $\left(\bar{u}_{pj}\right)_{opt} < MSE\ (\bar{u}_5)$, if

$$\lambda_0 \bar{Y}^2 C_y^2 \left(1 - \rho_{Y\pi_1}^2\right) - \bar{Y}^2 [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

$$\text{or } \lambda_0 C_y^2 \left(1 - \rho_{Y\pi_1}^2\right) - [A_4^2 F_{002} + A_3^2 F_{020} + F_{200} + 2(A_3 A_4 F_{011} - A_4 F_{101} - A_3 F_{110})] > 0$$

Condition (vii): MSE $\left(\bar{u}_{pj}\right)_{opt} < MSE\left(\bar{u}_6\right)$, if

$$\bar{Y}^2\left[C_Y^2+C_{\pi_1}^2+C_{\pi 2}^2-2\left(\rho_{yz_1}C_yC_{\pi_1}-\rho_{\pi 1\pi_2}C_{\pi_1}C_{\pi_2}+\rho_{y\pi_2}C_yC_{\pi_2}\right)\right]-$$
$$\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$
$$\text{or }\lambda_0\left[C_Y^2+C_{z_1}^2+C_{\pi 2}^2-2\left(\rho_{y\pi_1}C_yC_{\pi_1}-\rho_{\pi_1\pi_2}C_{z_1}C_{\pi_2}+\rho_{y\pi_2}C_yC_{\pi_1}\right)\right]-$$
$$\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$

Condition (viii): MSE $\left(\bar{u}_{pj}\right)_{opt} < MSE\left(\bar{u}_7\right)$, if

$$\lambda_0\bar{Y}^2\left[C_Y^2+C_{\pi_1}^2+C_{\pi 2}^2+2\left(\rho_{yx_1}C_yC_{x_1}-\rho_{x_1R_2}C_{x_1}C_{\pi_2}+\rho_{jz_2}C_yC_{x_2}\right)\right]-$$
$$\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$
$$\text{or }\lambda_0\left[C_Y^2+C_{\pi 1}^2+C_{\pi 2}^2+2\left(\rho_{j\pi_1}C_yC_{\pi_1}-\rho_{\pi 1}R_2C_{\pi 1}C_{\pi_2}+\rho_{j\pi_2}C_yC_{\pi_2}\right)\right]-$$
$$\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{100}\right)\right]>0$$

Condition (ix): MSE $\left(\bar{u}_{Pj}\right)_{apt} <$ MSE $\left(\bar{u}_8\right)$, if

$$\lambda_0\bar{Y}^2\left(C_Y^2+\frac{1}{4}C_{\pi_1}^2+\frac{1}{4}C_{\pi 2}^2-\rho_{yz_1}C_yC_{\pi_1}-\rho_{j\pi_2}C_yC_{\pi x_2}+\frac{1}{2}\rho_{\pi\pi_2}C_{\pi 1}C_{\pi 2}\right)-$$
$$\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$
$$\text{or }\lambda_0\left(C_Y^2+\frac{1}{4}C_{\pi 1}^2+\frac{1}{4}C_{\pi 2}^2-\rho_{y\pi_1}C_yC_{\pi_1}-\rho_{y\pi_2}C_yC_{\pi_2}+\frac{1}{2}\rho_{\pi x_2}C_{\pi 1}C_{\pi_2}\right)-$$
$$\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{01}-A_4F_{101}-A_3F_{101}\right)\right]>0$$

Condition (x): MSE $\left(\bar{u}_{Pj}\right)_{\text{opt}} <$ MSE $\left(\bar{u}_9\right)$, if

$$\lambda_0\bar{Y}^2\left(C_Y^2+\frac{1}{4}C_{\pi_1}^2+\frac{1}{4}C_{\pi 2}^2-\rho_{yx_1}C_yC_{\pi_1}+\rho_{y\pi_2}C_yC_{\pi_2}-\frac{1}{2}\rho_{\pi_{\pi_1}}C_{\pi_1}C_{\pi_2}\right)-$$
$$\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$
$$\text{or }\lambda_0\left(C_Y^2+\frac{1}{4}C_{\pi_1}^2+\frac{1}{4}C_{\pi 2}^2-\rho_{y\pi_1}C_yC_{\pi_1}+\rho_{yn_2}C_yC_{\pi_1}-\frac{1}{2}\rho_{\pi x_2}C_{\pi_1}C_{\pi_2}\right)-$$
$$\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{100}\right)\right]>0$$

Condition (xi): MSE $\left(\bar{u}_{Pj}\right)_{\text{ops}}$ < MSE $\left(\bar{u}_{10}\right)$, if

$$\lambda_0\bar{Y}^2\left[C_Y^2+\frac{1}{4}C_{\pi_1}^2+\frac{1}{4}\left(\frac{f}{1-f}\right)^2C_{\pi_2}^2-\rho_{Y\pi_1}C_YC_{\pi_1}-\left(\frac{f}{1-f}\right)\rho_{Y\pi_2}C_YC_{\pi_2}+\frac{1}{2}\left(\frac{f}{1-f}\right)\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2}\right]-$$
$$\bar{Y}^2\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$
$$\text{or }\lambda_0\left[C_Y^2+\frac{1}{4}C_{\pi_1}^2+\frac{1}{4}\left(\frac{f}{1-f}\right)^2C_{\pi_2}^2-\rho_{Y\pi_1}C_YC_{\pi_1}-\left(\frac{f}{1-f}\right)\rho_{Y\pi_2}C_YC_{\pi_2}+\frac{1}{2}\left(\frac{f}{1-f}\right)\rho_{\pi_1\pi_2}C_{\pi_1}C_{\pi_2}\right]-$$
$$\left[A_4^2F_{002}+A_3^2F_{020}+F_{200}+2\left(A_3A_4F_{011}-A_4F_{101}-A_3F_{110}\right)\right]>0$$

5. Numerical Study

To compare the efficiencies of traditional and suggested estimators, we use the following three sets of data.

Data1[Source: Ahmad et al. [1]]

Suppose that Y denotes approximate fish quantity that recreational marine fishermen captured in 1995, π_1 denote the proportion of fishes caught in 1993 more than 1000 and π_2 represent the proportion of fishes caught in 1994 greater than 2000.

$$N=69,\quad n=14,\quad \bar{Y}=4514.89,\quad T_1=0.73913,\quad T_2=0.55072,\quad C_{\pi_1}=0.59844,$$
$$C_{\pi_2}=0.90982,\quad C_Y=1.350,\quad \rho_{\pi_1\pi_2}=0.65775,\quad \rho_{Y\pi_2}=0.538074,\text{ and }\rho_{Y\pi_1}=0.39660$$

Data2 [Source: Ahmad et al. [1]]

Suppose area of tobacco production in hectares in 2009 denoted by Y, the proportion of farms with area in more than 500 hectares of tobacco cultivation in 2007 is denoted by π_1 and the proportion of farms with area for 47 districts of Pakistan of tobacco cultivation in 2007 is denoted by π_2.

$$N=47,\quad n=10,\quad \bar{Y}=1004.447,\quad T_1=0.42553,\quad T_2=0.38297,\quad C_{\pi_1}=1.17445,$$
$$C_{\pi 2}=1.28301,\quad C_Y=2.34124,\quad \rho_{\pi\pi_1\pi_2}=0.91538,\quad \rho_{\pi_{\pi_2}}=0.46615,\text{ and }\rho_{Y_{\pi_1}}=0.43959$$

Data 3: [Source: Malik and Singh [12]]

The population contains rice production areas of 73 districts in Pakistan. The detail of the variables is given below:

Y: production of rice in 2003.

π_1 : farms production where rice production is above 20 ton in 2002.

π_2 : farms production where rice production area is above 20 hectares in 2003.

$$N=73,\quad \bar{Y}=61.3,\quad T_1=0.42553,\quad T_2=0.38297,\quad S_Y^2=12371.4,$$
$$S_{\pi_1}^2=0.22549,\quad S_{\pi_2}^2=0.228311,\quad \rho_{\pi_1\pi_2}=0.889,\quad \rho_{\pi_{\pi_2}}=0.673,\text{ and }\rho_{Y_{\pi_1}}=0.621$$

The empirical values of MSEs, PREs and conditions (i)-(xi) are given in Table 1 and 2.

Estimators		Data 1		Data 2		Data 3	
		MSE	PRE	MSE	PRE	MSE	PRE
1	$\overline{\bar{u}_0}$	2115181	100.00	435346.4	100.00	317.8206	100.00
2	$\overline{u_1}$	1787091	18.3588	352892.1	123.3653	203.9578	155.8266
3	$\bar{u}_2$	3274557	64.5944	736889.4	59.07893	824.8486	38.53078
4	$\bar{u}_3$	1847225	114.5058	366733.2	118.7093	211.7435	150.0969
5	$\bar{u}_4$	2590958	81.63701	558731.8	77.91688	522.1889	60.86314
6	$\bar{u}_s$	1782481	118.665	351217.3	123.9536	195.2559	162.7713
7	$\bar{u}_6$	2045092	103.4272	480305.2	90.63954	526.3807	60.37846
8	$\bar{u}_7$	4937886	42.83575	870958	49.98478	1114.62	28.51380
9	$\bar{u}_8$	1528244	138.406	342980.8	126.9303	187.3936	169.6005
10	$\bar{u}_9$	3062169	69.07459	565403.3	76.99749	607.2162	52.34060
11	$\bar{u}_{10}$	1875375	112.7871	393837.3	110.5397	260.2787	122.1078
12	$\bar{u}_{P1}$	1496257	141.3648	340496.4	127.8564	173.9606	182.6969
13	$\bar{u}_{p1}$	1497943	141.2057	340630.1	127.8062	176.1853	180.3900
14	$\bar{u}_{p3}$	1497105	141.2847	340577.2	127.8261	174.4761	182.1570
15	$\bar{u}_{p4}$	1503779	140.6577	340765.2	127.7555	177.9408	178.6103

Table 1: Results of MSEs and PREs of different estimators using three data sets

5.1 Performance criteria

To evaluate the efficiency of proposed and existing estimators, two criteria are used. These are:

i. The mean square error of the estimators.

ii. The percentage relative efficiency (PRE) of the suggested and existing estimators.

The PRE of the suggested estimators $\bar{u}_{Pj}$, compared to existing estimators are computed by:

$$PRE = \frac{V\left(\bar{u}_0\right)}{\text{MSE}(r) \text{ or } \text{MSE}(r)_{\text{opt}}} \times 100,$$

$$r = \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5, \bar{u}_6, \bar{u}_7, \bar{u}_8, \bar{u}_9, \bar{u}_{10}, \bar{u}_{Pj}$$

5.2 Simulation Study

A simulation study is performed to find the MSE and PRE for evaluating the performance of suggested and competing estimators. Simulations of 15000 are performed using theoretical means μ = [4, 4, 4] and variances equal to 1 . The values of coefficient of correlations are given by:

$$\rho_{Y_{\pi_1}} = 0.69786, \rho_{Y_{\pi_2}} = 0.70643 \text{ and } \rho_{\pi_1\pi_2} = 0.90786$$

Samples of size n = 50, 90 and 160 are selected without replacement using simple random sampling. MSE and PRE are calculated and given in Table 3.

Conditions	Existing Estimators	Data	Proposed	Estimators		
			$\bar{u}_{p1}$	$\bar{u}_{P2}$	$\bar{u}_{P_3}$	$\bar{u}_{P4}$
i	$\bar{u}_0$	1	618924	617238	618076	611402
		2	94850.0	94716.3	94769.2	94581.2
		3	143.860	141.635	143.3445	139.8798
ii	$\bar{u}_1$	1	290834	289148	289986.0	283312
		2	12395.7	12262.0	12314.90	12126.9
		3	29.9972	27.7725	29.48170	26.0170
		1	1778300	1776614	1777452	1770778
iii	$\bar{u}_2$	2	396393	396259.3	396312.2	396124.2
		3	650.888	648.6633	650.3725	646.9078
		1	350968	349282.0	350120.0	343446.0
iv	$\bar{u}_3$	2	26236.8	26103.1	26156.00	25968.0
		3	37.7829	35.5582	37.26740	33.80270
		1	1094701	1093015	1093853.0	1087179
v	$\bar{u}_4$	2	218235.4	218101.7	218154.6	217966.6
		3	348.2283	346.0036	347.7128	344.2481
vi		1	286224.0	284538.0	285376.0	278702.0
	$\bar{u}_5$	2	10720.90	10587.20	10640.10	10452.10
		3	21.29530	19.07060	20.77980	17.31510
vii		1	548835.0	547149.0	547987.0	541313.0
	$\bar{u}_6$	2	139808.8	139675.1	139728.0	139540.0
		3	352.4201	350.1954	351.9046	348.4399
viii		1	3441629	3439943	3440781	3434107
	$\bar{u}_7$	2	530461.6	530327.9	530380.8	530192.8
		3	940.6594	938.4347	940.1439	936.6792
ix		1	31987.00	30301.00	31139.0	24465.00
	$\bar{u}_8$	2	2484.4 .00	2350.700	2403.60	2215.600
		3	13.43300	11.20830	12.9175	9.452800
x		1	1565912	1564226	1565064	1558390
	$\bar{u}_9$	2	224906.9	224773.2	224826.1	224638.1
		3	433.2556	431.0309	432.7401	429.2754
xi		1	379118.0	377432.0	3782700	371596.0
	$\bar{u}_{10}$	2	53340.90	53207.20	53260.10	53072.10
		3	86.31810	84.09340	85.8026	82.33790

Table 2: Numerical verification of conditions (i)-(xi)

Estimators		n = 50		n = 90		n = 160	
		MSE	PRE	MSE	PRE	MSE	PRE
1	$\bar{u}_0$	3263274	100	236846.5	100	298.978	100
2	$\bar{u}_1$	11660088.54	27.9867	187345.95	126.422	180.700	165.455
3	$\bar{u}_2$	4239339.53	76.976	349979.97	67.6743	526.694	56.765
4	$\bar{u}_3$	2749740.68	118.676	188010.716	125.975	191.533	156.097
5	$\bar{u}_4$	3823308.39	85.3521	275584.92	85.9432	449.005	66.5867
6	$\bar{u}_5$	2634770.561	123.854	183111.94	129.345	177.133	168.787
7	$\bar{u}_6$	3074076.86	106.155	248417.54	95.3421	468.702	63.7885
8	$\bar{u}_7$	6957435.92	46.9034	423790.79	55.8876	833.353	35.8765
9	$\bar{u}_8$	2266326.18	143.99	186153.83	127.232	170.2162	175.646
10	$\bar{u}_9$	4241642.86	76.9342	295189.16	80.2355	507.918	58.8634
11	$\bar{u}_{10}$	2802710.23	116.433	209621.67	112.988	224.971	132.896
12	$\bar{u}_{P1}$	2240728.67	145.635	180848.55	130.964	160.850	185.873
13	$\bar{u}_{p2}$	2251210.71	144.956	182530.95	129.757	160.082	186.765
14	$\bar{u}_{p3}$	2328842.32	140.124	182514.63	129.769	160.848	185.876
15	$\bar{u}_{p_4}$	2300406.61	141.856	183809.54	128.854	164.403	181.856

Table 3: MSEs and PREs of different estimators using simulation study.

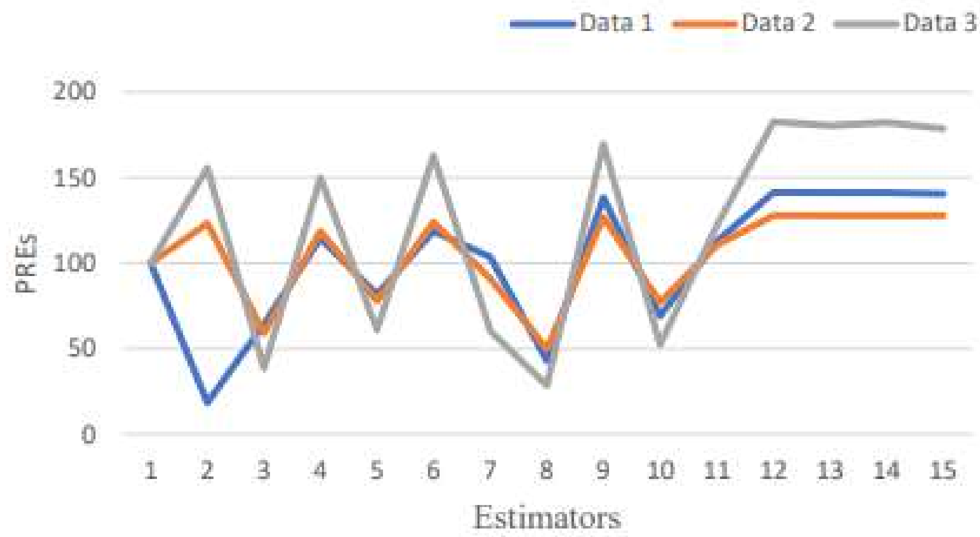

Figure 1. Figure: PREs of existing and proposed estimators for all three data sets.

6. Results and Discussion

In this paper, a family of improved mean estimators for population using supplementary attributes under simple random sampling is suggested. To evaluate the performance of the proposed estimators, we used three real data sets and simulation studies. The criteria of MSE and PRE are used for comparison of different estimators. The MSEs and PREs of the suggested and existing estimators are provided in Table 1 for the real data sets and in Table 3, the values of the simulation studies are given. Verification's of numerical conditions are provided in Table 2. PREs of the proposed and existing estimators are also presented through a figure for real data sets. The results of the proposed family of estimators vary, based on different choices of $\omega_1, \omega_2, \omega_3$ and ω_4.

The following are some general findings as:

- (i) Table 1 reveals that all the suggested estimators $\bar{u}_{pj}$ have the optimum MSE over all the competitor estimators for all three data sets. This verify that the suggested estimators $\bar{u}_{Pj}$ perform the best than the competitor estimators.
- (ii) The proposed estimator $\bar{u}_{P1}$ has the minimum MSE for $\omega_1 = C_{\pi_2}, \omega_2 = T_2, \omega_3 = \rho^3_{Y_{\pi_1}}$, and $\omega_4 = C^3_{\pi_2}$, among all the proposed estimators for all the real data sets.
- (iii) Table 2 reveals that all the numerical differences are greater than zero which verify the efficiency conditions theoretically described in Section 4.
- (iv) From the Figure of real data sets, it is observed that there is maximum PRE for all the suggested estimators as compared to the existing estimators. The lines in the figure is in upward direction and having values above than the values of existing values.
- (v) The applications of the suggested estimators are also demonstrated in Table 3, using simulation studies. The results reveal that all four suggested estimators have minimum MSE as compared to

competitor estimators.

- (vi) Overall, the suggested estimator $\bar{u}_{p1}$ has shown the best performance among all the proposed and existing estimators considered in the study for both real data sets and simulation study.

7. Conclusion

An improved family of estimators is suggested for the finite population mean using supplementary attributes. Bias and mean square error of the proposed estimators are derived theoretically up to the first-order approximation and compared with the competitor estimators.The suggested estimators are also compared with the existing estimators numerically and graphically using PRE and the MSE criteria to support the theoretical results. For numerical comparison, three real data sets and simulation study are used. Table 1,2 and 3 provides the numerical results which reveals that the proposed estimators $\bar{u}_{pj}$ show the best results as compared to their counterparts.The numerical results also vindicated the efficiency conditions of the suggested estimators, which is described theoretically in section 4. In consequent, we recommend our proposed estimators for the new survey to obtain better and more efficient results instead of the usual estimators examine in this article for estimating the finite population mean using supplementary attributes.

Recommendations:

It is recommended that this study can be extended by developing new estimators with the help of more selections of $\omega_1, \omega_2, \omega_3$ and ω_4.

Data Availability

The data used to support the findings of the study are available within this article.

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Author Contributions

Mujeeb Hussain: Conceptualization, Supervision, analyzed and interpreted data. **Qamruz Zaman**: Data curation, Methodology. **Lakhkar Khan**: Investigation, software, removed all the grammatical mistakesv. **Abdurrahman Sabir**: Improved language, Editing.

Compliance with Ethical Standards

The authors declare that they have no conflicts of interest. Additionally, this article does not involve with human participants or animals conducted by any of the authors. Furthermore, informed consent was taken from all individual participants involved in the study.

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